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The Many Faces of Gravitoelectromagnetism

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Abstract

The numerous ways of introducing spatial gravitational forces are fit together in a single framework enabling their interrelationships to be clarified. This framework is then used to treat the "acceleration equals force" equation and gyroscope precession, both of which are then discussed in the post-Newtonian approximation, followed by a brief examination of the Einstein equations themselves in that approximation.

1 Introduction

The concept of spatial gravitational forces modeled after the electromagnetic Lorentz force has a long history and many names associated with it [1]–[17]. Born in the Newtonian context of centrifugal and Coriolis forces introduced by a rigidly rotating coordinate system in a flat Euclidean space, it has found a number of closely related but distinct generalizations within the context of general relativity and its linearized approximation. With the frequent reference to "gravitoelectromagnetism" occurring in recent literature, it is time to place

all of these notions of "noninertial forces" into a single framework which in turn may be used to infer relationships among them.

Key to all of these notions is the splitting of spacetime into "space plus time", accomplished locally by means of an observer congruence, namely a congruence of timelike worldlines with (future-pointing) unit tangent vector field u which may be interpreted as the 4-velocity field of a family of test observers filling the spacetime or some open submanifold of it [18]–[23]. These worldlines have a natural parametrization by the proper time τ_u measured along them and defined to within an initial value on each worldline. The orthogonal decomposition of each tangent space into a local time direction along u and the orthogonal local rest space LRS_u may be used to decompose all spacetime tensors and tensor equations into a "space plus time" representation, i.e., to "measure" them. This leads to a family of "spatial" spacetime tensor fields (giving zero upon any contraction with u) which represent each spacetime field and a family of spatial equations which represent each spacetime equation.

Such a splitting permits a better interface of our 3-dimensional intuition and experience with the 4-dimensional spacetime geometry in certain gravitational problems, though it may complicate others. It can be particularly useful in spacetimes which have a geometrically defined timelike congruence, either explicitly given or defined implicitly as the congruence of orthogonal trajectories to a slicing (foliation) of spacetime by a family of spacelike hypersurfaces, the latter leading to a timelike congruence with vanishing vorticity or "rotation" (the hypersurface-forming condition for the distribution LRS_u). Stationary spacetimes have a preferred congruence of Killing trajectories associated with the stationary symmetry, which is timelike on an open submanifold of spacetime. Stationary axially symmetric spacetimes have in addition a preferred slicing whose orthogonal trajectories coincide with the worldlines of locally nonrotating test observers on an open submanifold of spacetime [24]–[26]. Cosmological spacetimes with a spatial homogeneity subgroup have a preferred spacelike slicing by the orbits of this subgroup.

A partial splitting of spacetime based only on a timelike congruence (splitting off time alone) or a spacelike slicing (splitting off space alone) will be referred to as the congruence and hypersurface splittings respectively. Often a congruence and transversal slicing occur in the same context, with at least one of the components satisfying the causality condition of the corresponding splitting. Such a pair will be said to define a "nonlinear reference frame" (to avoid confusion with existing terms) and a full splitting of spacetime into "time plus space" $(1 + 3)$ or "space plus time" $(3 + 1)$ respectively [27]–[43]. Introduce the suggestive term "threading" parallel to the term "slicing," in order to describe the transversal congruence which "threads" (by transversality) the slicing. The two full splittings will be called the threading splitting (timelike threading) and the slicing splitting (spacelike slicing). Each has an associated observer congruence of the corresponding partial splitting. When both causality conditions hold, both splittings are valid and one may transform between them, unless the nonlinear reference frame is orthogonal (orthogonal slicing and threading), in which case they coincide. The nonlinear reference frame itself provides another

splitting which is often used to represent the former two, namely the (in general) nonorthogonal splitting of the tangent spaces into the local threading direction and the local slicing directions. This will be called the reference splitting.

In addition, the threading or slicing may be provided with a parametrization, namely a class of affinely related parameters on each congruence curve or of the family of slices respectively. In a parametrized nonlinear reference frame, both components may be compatibly parametrized, with their parametrizations linked in an obvious way. In a stationary spacetime the canonical parameter on the orbits of the stationary symmetry provides a natural parametrization for the timelike Killing threading, while in a spatially homogeneous spacetime the proper time measured orthogonally to the family of geodesically parallel spatial hypersurfaces of homogeneity is a natural parameter for that preferred slicing.

Spatial gravitational forces have been defined in all of these contexts, depending on or independent of the parametrizations, both in the fully nonlinear theory as well as in the linearized theory. The proper question to ask is not which of these various descriptions to choose is the "best" or "correct" one, but what exactly each one of them measures and which is particularly suited to a particular application where it can help provide intuition about or simplify the presentation of the invariant spacetime geometry that all of them may be used to reconstruct. Until now there has been no effort to clarify the interrelationships between the many different approaches favored by numerous groups working with isolated formalisms. A true relativity of formalism is needed to break the barrier to a more versatile application of multiple approaches whose selection is determined by the application and not by the inertia of the investigator. A careful development of this relativity of formalism, as well as an appropriate historical survey of the topic, requires a more lengthy exposition [44], so only a brief sketch will be presented here, limiting historical credit to references in the text.

The slicing point of view, often called the ADM approach [29], has been effectively promoted by the textbook by Misner, Thorne and Wheeler [30], whose conventions will be assumed unless otherwise indicated. The same effective notation and terminology will be extended to the threading point of view, partially presented in the textbook by Landau and Lifshitz [2].

2 Observer-orthogonal splitting

Let ${}^{(4)}\mathrm{g}$ (signature -+++ and components ${}^{(4)}g_{\alpha\beta}$, $\alpha, \beta, \ldots = 0,1,2,3$) be the spacetime metric, ${}^{(4)}\nabla$ its associated covariant derivative operator, and ${}^{(4)}\eta$ the unit volume 4-form which orients spacetime (${}^{(4)}\eta_{0123} = {}^{(4)}g^{1/2}$ in an oriented frame, where ${}^{(4)}g \equiv |\det({}^{(4)}g_{\alpha\beta})|$). Assume the spacetime is also time oriented and let u be a future-pointing unit timelike vector field ($u^\alpha u_\alpha = -1$) representing the 4-velocity field of a family of test observers filling the spacetime (or some open submanifold of it). If S is an arbitrary tensor field, let $S^\flat$ and $S^\sharp$ denote its totally covariant and totally contravariant forms with respect to the metric index-shifting operations. It is also convenient to introduce the right contraction

notation $[S \llcorner X]^\alpha = S^\alpha{}_\beta X^\beta$ for the contraction of a vector field and the covariant index of a $\binom{1}{1}$-tensor field, representing the action of a linear transformation of each tangent space into itself. In general let the left contraction $S \lrcorner T$ denote the tensor product of the two tensors S and T with a contraction between the rightmost contravariant index of S with the leftmost covariant index of T (i.e., $S^{\cdots\alpha}{}_{\cdots} T^{\cdots}{}_{\alpha\ldots}$), and let the right contraction $S \llcorner T$ denote the tensor product with a contraction between the leftmost contravariant index of T with the rightmost covariant index of S (i.e., $S^{\cdots}{}_{\ldots\alpha} T^{\alpha\cdots}{}_{\ldots}$), assuming in each case that such indices exist. For a $\binom{1}{1}$-tensor field S, let $S^2 \equiv S \llcorner S$.

The observer-orthogonal decomposition of the tangent space, and in turn of the algebra of spacetime tensor fields, is accomplished by the temporal projection operator $T(u)$ along u and the spatial projection operator $P(u)$ onto LRS_u, which may be identified with mixed second rank tensors acting by contraction

$$\begin{aligned} \delta^\alpha{}_\beta &= T(u)^\alpha{}_\beta + P(u)^\alpha{}_\beta \ , \\ T(u)^\alpha{}_\beta &= -u^\alpha u_\beta \ , \\ P(u)^\alpha{}_\beta &= \delta^\alpha{}_\beta + u^\alpha u_\beta \ . \end{aligned} \tag{2.1}$$

These satisfy the usual orthogonal projection relations $P(u)^2 = P(u)$, $T(u)^2 = T(u)$, and $T(u) \llcorner P(u) = P(u) \llcorner T(u) = 0$. Let

$$[P(u)S]^{\alpha\cdots}{}_{\beta\ldots} = P(u)^\alpha{}_\gamma \cdots P(u)^\delta{}_\beta \cdots S^{\gamma\cdots}{}_{\delta\ldots} \tag{2.2}$$

denote the spatial projection of a tensor S on all indices.

The "measurement of S" by the observer congruence is the family of spatial tensor fields which result from the spatial projection of all possible contractions of S by any number of factors of u. For example, if S is a $\binom{1}{1}$-tensor, then its measurement

$$S^\alpha{}_\beta \leftrightarrow (u^\delta u_\gamma S^\gamma{}_\delta, P(u)^\alpha{}_\gamma u^\delta S^\gamma{}_\delta, P(u)^\delta{}_\alpha u_\gamma S^\gamma{}_\delta, P(u)^\alpha{}_\gamma P(u)^\delta{}_\beta S^\gamma{}_\delta) \tag{2.3}$$

results in a scalar field, a spatial vector field, a spatial 1-form and a spatial $\binom{1}{1}$-tensor field. It is exactly this family of fields which occur in the (orthogonal) "decomposition of S" with respect to the observer congruence

$$\begin{aligned} S^\alpha{}_\beta &= [T(u)^\alpha{}_\gamma + P(u)^\alpha{}_\gamma][T(u)^\delta{}_\beta + P(u)^\delta{}_\beta] S^\gamma{}_\delta \\ &= [u^\delta u_\gamma S^\gamma{}_\delta] u^\alpha u_\beta + \cdots + [P(u)S]^\alpha{}_\beta \ . \end{aligned} \tag{2.4}$$

The spatial metric $[P(u)^{(4)}\mathrm{g}]_{\alpha\beta} = P(u)_{\alpha\beta}$ and the spatial unit volume 3-form $\eta(u)_{\alpha\beta\gamma} = u^{\delta\,(4)}\eta_{\delta\alpha\beta\gamma} = [P(u)\, u \lrcorner {}^{(4)}\eta]_{\alpha\beta\gamma}$ are the only nontrivial spatial fields which result from the measurement of the spacetime metric and volume 4-form.

Introduce also the spatial Lie derivative [34] $\pounds(u)_X = P(u)\pounds_X$ by the vector field X, the spatial exterior derivative $d(u) = P(u)d$, the spatial covariant derivative $\nabla(u) = P(u)^{(4)}\nabla$, the spatial Fermi-Walker derivative ("Fermi-Walker temporal derivative") $\nabla_{(\mathrm{fw})}(u) = P(u)^{(4)}\nabla_u$ and the Lie temporal derivative $\nabla_{(\mathrm{lie})}(u) = P(u)\pounds_u = \pounds(u)_u$. Note that these spatial differential

operators do not obey the usual product rules for nonspatial fields since undifferentiated factors of u are killed by the spatial projection.

It is convenient to introduce 3-dimensional vector notation for the spatial inner product and spatial cross product of two spatial vector fields X and Y. The inner product is just

$$X \cdot_u Y = P(u)_{\alpha\beta} X^\alpha Y^\beta \tag{2.5}$$

while the cross product is

$$[X \times_u Y]^\alpha = \eta(u)^\alpha{}_{\beta\gamma} X^\beta Y^\gamma \ . \tag{2.6}$$

If one lets $\vec{\nabla}(u)$ be the "vector derivative operator" $\nabla(u)^\alpha$, then one can introduce spatial gradient, curl and divergence operators for functions f and spatial vector fields X by

$$\begin{aligned} \mathrm{grad}_u\, f &= \vec{\nabla}(u) f = [d(u)f]^\sharp \ , \\ \mathrm{curl}_u\, X &= \vec{\nabla}(u) \times_u X = [{}^{*(u)} d(u) X^\flat]^\sharp \ , \\ \mathrm{div}_u\, X &= \vec{\nabla}(u) \cdot_u X = {}^{*(u)}[d(u) {}^{*(u)} X^\flat] \ , \end{aligned} \tag{2.7}$$

where ${}^{*(u)}$ is the spatial duality operation for antisymmetric tensor fields associated with the spatial volume form $\eta(u)$ in the usual way. These definitions enable one to mimic all the usual formulas of 3-dimensional vector analysis. The spatial exterior derivative formula for the curl has the index form

$$[\mathrm{curl}_u\, X]^\alpha = \eta(u)^{\alpha\beta\gamma\,(4)}\nabla_\beta X_\gamma \tag{2.8}$$

and also defines a useful operator for nonspatial vector fields X.

Measurement of the covariant derivative $[{}^{(4)}\nabla u]^\alpha{}_\beta = u^\alpha{}_{;\beta}$ leads to two spatial fields, the acceleration vector field $a(u)$ and the kinematical mixed tensor field $k(u)$

$$\begin{aligned} u^\alpha{}_{;\beta} &= -a(u)^\alpha u_\beta - k(u)^\alpha{}_\beta \ , \\ a(u) &= \nabla_{(\mathrm{fw})}(u) u \ , \\ k(u) &= -\nabla(u) u = \omega(u) - \theta(u) \ . \end{aligned} \tag{2.9}$$

The kinematical tensor field may be decomposed into its antisymmetric and symmetric parts [18]–[21, 30, 45]

$$\begin{aligned} [\omega(u)^\flat]_{\alpha\beta} &= P(u)^\delta{}_\beta P(u)^\gamma{}_\alpha u_{[\delta;\gamma]} = \tfrac{1}{2}[d(u)u^\flat]_{\alpha\beta} \ , \\ [\theta(u)^\flat]_{\alpha\beta} &= P(u)^\delta{}_\beta P(u)^\gamma{}_\alpha u_{(\beta;\alpha)} = \tfrac{1}{2}[\nabla_{(\mathrm{lie})}(u) P(u)^\flat]_{\alpha\beta} = \tfrac{1}{2}\pounds(u) u^{(4)} g_{\alpha\beta} \ , \end{aligned} \tag{2.10}$$

defining the mixed rotation or vorticity tensor field $\omega(u)$ (whose sign depends on convention) and the mixed expansion tensor field $\theta(u)$, the latter of which may itself be decomposed into its tracefree and pure trace parts

$$\theta(u) = \sigma(u) + \frac{1}{3}\Theta(u) P(u) \ , \tag{2.11}$$

where the mixed shear tensor field $\sigma(u)$ is tracefree ($\sigma(u)^\alpha{}_\alpha = 0$) and the expansion scalar is

$$\Theta(u) = u^\alpha{}_{;\alpha} = {}^{*(u)}[\nabla_{(\mathrm{lie})}(u)\eta(u)] \ . \tag{2.12}$$

Define also the rotation or vorticity vector field $\vec{\omega}(u) = \frac{1}{2}\,\mathrm{curl}_u\, u$ as the spatial dual of the spatial rotation tensor field

$$\omega(u)^\alpha = \tfrac{1}{2}\eta(u)^{\alpha\beta\gamma}\omega(u)_{\beta\gamma} = \tfrac{1}{2}{}^{(4)}\eta^{\alpha\beta\gamma\delta}u_\beta u_{\gamma;\delta} \ . \tag{2.13}$$

The kinematical tensor describes the difference between the Lie and Fermi-Walker temporal derivative operators when acting on spatial tensor fields. For example, for a spatial vector field X

$$\begin{aligned}\nabla_{(\mathrm{fw})}(u)X^\alpha &= \nabla_{(\mathrm{lie})}(u)X^\alpha - k(u)^\alpha{}_\beta X^\beta \\ &= \nabla_{(\mathrm{lie})}(u)X^\alpha - \omega(u)^\alpha{}_\beta X^\beta + \theta(u)^\alpha{}_\beta X^\beta \ ,\end{aligned} \tag{2.14}$$

where

$$\omega(u)^\alpha{}_\beta X^\beta = -\eta(u)^\alpha{}_{\beta\gamma}\omega(u)^\beta X^\gamma = -[\vec{\omega}(u) \times_u X]^\alpha \ . \tag{2.15}$$

Spatial vector fields which undergo spatial Lie transport along u, i.e., $\nabla_{(\mathrm{lie})}(u)X = 0$, are called "connecting vectors" since they have the interpretation of being the relative position vectors of nearby observers in the limit of vanishingly small magnitude. This equation shows how such connecting vector fields change along u with respect to a spatial Fermi-Walker transported spatial frame along u, giving the usual physical interpretation of the individual kinematical fields. Apart from shear and expansion effects, the Fermi-Walker transported spatial vectors have an angular velocity $-\vec{\omega}(u)$ with respect to spatial vectors undergoing spatial Lie transport along u, or conversely the connecting vectors rotate with angular velocity $\vec{\omega}(u)$ with respect to an orthonormal spatial frame which is Fermi-Walker transported along u.

The kinematical quantities associated with u may be used to introduce two spacetime temporal derivatives, the Fermi-Walker derivative [30, 46, 47] and the co-rotating Fermi-Walker derivative [48] along u

$$\begin{aligned}{}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha &= {}^{(4)}\nabla_u X^\alpha + [a(u)\wedge u]^{\alpha\beta}X_\beta \ , \\ {}^{(4)}\nabla_{(\mathrm{cfw})}(u)X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha + \omega(u)^\alpha{}_\beta X^\beta \ .\end{aligned} \tag{2.16}$$

These may be extended to arbitrary tensor fields in the usual way (so that they commute with contraction and tensor products) and they both commute with index shifting with respect to the metric and with duality operations on antisymmetric tensor fields since both ${}^{(4)}\mathrm{g}$ and ${}^{(4)}\eta$ have zero derivative with respect to both operators (as does u itself). An arbitrary tensor field for which one of these operators yields zero will be said to undergo respectively either Fermi-Walker transport along u or co-rotating Fermi-Walker transport along u. The Fermi-Walker transport differs from parallel transport by a boost in the plane of u and $a(u)$ which maps the parallel transport of u onto u itself. The co-rotating

Fermi-Walker transport differs by an additional rotation in LRS_u which causes it to co-rotate with the observer congruence, i.e., to remain constant with respect to a spatial orthonormal frame undergoing this transport, the individual frame vectors of which co-rotate with respect to nearby observers, without undergoing the shear and expansion of the connecting vectors. These both differ from Lie transport along u in the following manner

$$\begin{aligned} \pounds_u X^\alpha &= {}^{(4)}\nabla_{(\mathrm{fw})}(u)X^\alpha + [\omega(u)^\alpha{}_\beta - \theta(u)^\alpha{}_\beta + u^\alpha a(u)_\beta]X^\beta \\ &= {}^{(4)}\nabla_{(\mathrm{cfw})}(u)X^\alpha + [-\theta(u)^\alpha{}_\beta + u^\alpha a(u)_\beta]X^\beta \ . \end{aligned} \tag{2.17}$$

A spatial co-rotating Fermi-Walker derivative $\nabla_{(\mathrm{cfw})}(u)$ ("co-rotating Fermi-Walker temporal derivative") may be defined in a way analogous to the ordinary one, such that the three temporal derivatives have the following relation when acting on a spatial vector field X

$$\begin{aligned} \nabla_{(\mathrm{cfw})}(u)X^\alpha &= \nabla_{(\mathrm{fw})}(u)X^\alpha + \omega(u)^\alpha{}_\beta X^\beta \\ &= \nabla_{(\mathrm{lie})}(u)X^\alpha + \theta(u)^\alpha{}_\beta X^\beta \ , \end{aligned} \tag{2.18}$$

while $\nabla_{(\mathrm{cfw})}(u)[fu] = f a(u)$ determines its action on nonspatial fields. It is convenient to use an index notation to handle these three operators simultaneously

$$\{\nabla_{(\mathrm{tem})}(u)\}_{\mathrm{tem=fw,cfw,lie}} = \{\nabla_{(\mathrm{fw})}(u), \nabla_{(\mathrm{cfw})}(u), \nabla_{(\mathrm{lie})}(u)\} \ . \tag{2.19}$$

The Lie temporal derivative does not commute with index shifting of spatial fields by the metric or with the spatial duality operation using $\eta(u)$ but generates additional expansion tensor terms. Only the other two temporal derivatives are in general compatible with imposing an orthonormality condition on a spatial frame which undergoes their corresponding transport along u.

The restriction of the spatial Fermi-Walker derivative to purely spatial tensor fields is the derivative first introduced by Fermi [46]. The measurement of the ordinary or co-rotating Fermi-Walker derivative of an arbitrary tensor field results in the corresponding spatial derivative acting on each spatial tensor field of the collection of fields which represent the undifferentiated tensor field.

3 Observer-adapted frames

Components with respect to a frame adapted to the observer orthogonal decomposition can be quite useful in the splitting game, especially in splitting tensor fields with many indices. An "observer-adapted frame" $\{e_\alpha\}$ with dual frame $\{\omega^\alpha\}$ will be any frame for which e_0 is along u and the "spatial frame" $\{e_a\}_{a=1,2,3}$ spans the local rest space LRS_u

$$\begin{aligned} &u = L^{-1}e_0 \equiv e_\top \ , &&u^\flat(e_a) = 0 \ , \\ &u^\flat = -L\,\omega^0 \equiv -\omega^\top \ , &&\omega^a(u) = 0 \ . \end{aligned} \tag{3.1}$$

If it is oriented and time-oriented, then $L > 0$ and $\eta(u)_{123} > 0$. The index "$\top$" (pronounced "tan") suggests "tangential" to the congruence (or "temporal")

and corresponds to the orthonormal temporal component obtained by scaling the zero-indexed frame component by the normalization factor L. Similarly it is customary to use the index "$\perp$" ("perp") in the hypersurface point of view where u is perpendicular to the integrable distribution of local rest spaces.

The splitting of a tensor field S amounts to a partitioning of the components in an observer-adapted frame according to whether or not individual indices are zero or not. The purely spatial part corresponds to those components which have only "spatial indices" 1,2,3, i.e., no "temporal index" 0. For a $\binom{1}{1}$-tensor S one has

$$S \quad \leftrightarrow \quad \{S^0{}_0, S^a{}_0, S^0{}_a, S^a{}_b\} \ . \tag{3.2}$$

Rescaling each 0 index by an appropriate factor of L corresponds to the measurement process described above, apart from the sign difference between $u^\flat$ and $\omega^\top$. Spatial tensors have only the spatially-indexed components nonzero, so indexed formulas with Greek indices involving only spatial fields reduce to Latin-indexed formulas when expressed in an observer-adapted frame.

The spacetime metric and its inverse in such a frame have the form

$$\begin{aligned} {}^{(4)}\mathrm{g} &= -L^2\omega^0 \otimes \omega^0 + h_{ab}\omega^a \otimes \omega^b = -\omega^\top \otimes \omega^\top + h_{ab}\omega^a \otimes \omega^b \ , \\ {}^{(4)}\mathrm{g}^{-1} &= -L^{-2}e_0 \otimes e_0 + h^{ab}e_a \otimes e_b \ = -e_\top \otimes e_\top + h^{ab}e_a \otimes e_b \ , \end{aligned} \tag{3.3}$$

where (h_{ab}) is a positive-definite matrix with positive determinant h. The spacetime metric determinant factor has the expression ${}^{(4)}g^{1/2} = Lh^{1/2}$, while the oriented spatial volume 3-form has components $\eta(u)_{abc} = {}^{(4)}\eta_{\top abc} = h^{1/2}\epsilon_{abc}$. The spatial metric and its inverse are the covariant and contravariant forms of the spatial projection $P(u) = e_a \otimes \omega^b$

$$P(u)^\flat = h_{ab}\omega^a \otimes \omega^b \ , \quad P(u)^\sharp = h^{ab}e_a \otimes e_b \ . \tag{3.4}$$

One can also introduce the components of the spatial part of the spatial connection in an observer-adapted frame by making the usual definition

$$\nabla(u)_{e_a} e_b = \Gamma(u)^c{}_{ab} e_c \ . \tag{3.5}$$

Introducing several notations $\partial_\alpha f = f_{,\alpha} = e_\alpha f$ for the frame derivatives of functions, and the anticyclic permutation notation

$$A_{\{abc\}_-} = A_{abc} - A_{bca} + A_{cab} \ , \tag{3.6}$$

one finds the usual formula

$$\Gamma(u)_{abc} = \tfrac{1}{2}[h_{\{ab,c\}_-} + C(u)_{\{abc\}_-}] \ , \tag{3.7}$$

where $C(u)^a{}_{bc} = C^a{}_{bc} = \omega^a([e_b, e_c])$ are the spatial components of the Lie bracket tensor of this spatial frame with its indices shifted from the normal positions using the spatial metric. One then has familiar formulas like

$$[\nabla(u)_X Y]^a = X^b \nabla(u)_b Y^a = X^b [Y^a{}_{,b} + \Gamma(u)^a{}_{bc} Y^c] \tag{3.8}$$

for two spatial vector fields X and Y.

Of the remaining structure functions $C^{\alpha}{}_{\beta\gamma} = \omega^{\alpha}([e_\beta, e_\gamma])$ of the observer-adapted frame, some are closely related to the acceleration and rotation of u, while the remaining ones appear in the temporal Lie derivative of a spatial quantity, as in

$$\nabla_{(\mathrm{lie})}(u)X^a = L^{-1}[\partial_0 X^a + C^a{}_{0b}X^b] \ . \tag{3.9}$$

This in turn leads to explicit expressions for $\nabla_{(\mathrm{fw})}(u)X$ and $\nabla_{(\mathrm{cfw})}(u)X$ by Eq. (2.18). In particular

$$\nabla_{(\mathrm{tem})}(u)e_a = C_{(\mathrm{tem})}(u)^b{}_a e_b \ , \tag{3.10}$$

where

$$\begin{aligned} C_{(\mathrm{lie})}(u)^b{}_a &= L^{-1}C^b{}_{0a} \ , \\ C_{(\mathrm{cfw})}(u)^b{}_a &= L^{-1}C^b{}_{0a} + \theta(u)^b{}_a \ , \\ C_{(\mathrm{fw})}(u)^b{}_a &= L^{-1}C^b{}_{0a} + \theta(u)^b{}_a - \omega(u)^b{}_a \ , \end{aligned} \tag{3.11}$$

indicates three useful choices for fixing the otherwise arbitrary structure functions $C^b{}_{0a}$ which determine how the spatial frame is transported along u. Setting the matrix $C_{(\mathrm{tem})}(u)^a{}_b$ to zero for each of the three choices in turn respectively defines the spatial frame's spatial Lie transport, its co-rotating Fermi-Walker transport, and its Fermi-Walker transport along u.

4 Relative kinematics: algebra

Suppose U is another unit timelike vector field representing a different family of test observers. One can then consider relating the "observations" of each to the other. Their relative velocities are defined by

$$U = \gamma(U,u)[u + \nu(U,u)] \ , \qquad u = \gamma(u,U)[U + \nu(u,U)] \ , \tag{4.1}$$

where the relative velocity $\nu(U,u)$ of U with respect to u is spatial with respect to u and vice versa, both of which have the same magnitude $||\nu(U,u)|| = [\nu(U,u)_\alpha \nu(U,u)^\alpha]^{1/2}$, while the common gamma factor is related to that magnitude by

$$\gamma(U,u) = \gamma(u,U) = [1 - ||\nu(U,u)||^2]^{-1/2} = -U_\alpha u^\alpha \ . \tag{4.2}$$

Let $\hat\nu(U,u)$ be the unit vector giving the direction of the relative velocity $\nu(U,u)$.

Introduce also the energy and spatial momentum per unit mass relative to u

$$\tilde{E}(U,u) = \gamma(U,u) \ , \quad \tilde{p}(U,u) = \gamma(U,u)\nu(U,u) \ . \tag{4.3}$$

In addition to the natural parametrization of the worldlines of U by the proper time τ_U, one may introduce a new parametrization $\tau_{(U,u)}$ by

$$d\tau_{(U,u)}/d\tau_U = \gamma(U,u) \ , \tag{4.4}$$

which corresponds to the sequence of proper times of the family of observers from the u congruence which cross paths with a given worldline of the U congruence. It is convenient to abbreviate $\gamma(U,u)$ by γ when its meaning is clear from the context.

Eqs. (4.1) describe a unique active "relative observer boost" $B(U,u)$ in the "relative observer plane" spanned by u and U such that

$$B(U,u)u = U \ , \quad B(U,u)\nu(U,u) = -\nu(u,U) \tag{4.5}$$

and which acts as the identity on the common subspace of the local rest spaces $LRS_u \cap LRS_U$ orthogonal to the direction of motion. The inverse boost $B(u,U)$ "brings U to rest" relative to u. It will be convenient to use the same symbol for a linear map of the tangent space into itself and the corresponding $\binom{1}{1}$-tensor acting by contraction. The right contraction between two such maps will represent their composition. When the contraction symbol is suppressed, the linear map will be implied.

The projection $P(U)$ restricts to an invertible map $P(U,u) = P(U) \circ P(u) : LRS_u \to LRS_U$ with inverse $P(U,u)^{-1} : LRS_U \to LRS_u$ and vice versa, and these maps also act as the identity on the common subspace of the local rest spaces. Similarly the boost $B(U,u)$ restricts to an invertible map $B_{(\mathrm{lrs})}(U,u) \equiv P(U) \circ B(U,u) \circ P(u)$ between the local rest spaces which also acts as the identity on their common subspace. The boosts and projections between the local rest spaces differ only by a gamma factor along the direction of motion. It is exactly the inverse projection map which describes Lorentz contraction of lengths along the direction of motion. Figure 1 illustrates these maps on the relative observer plane of u and U.

If $Y \in LRS_u$, then the orthogonality condition $0 = u_\alpha Y^\alpha$ implies that Y has the form

$$Y = [\nu(u,U) \cdot_U P(U,u)Y]U + P(U,u)Y \ . \tag{4.6}$$

If $X = P(U,u)Y \in LRS_U$ is the field seen by U, then $Y = P(U,u)^{-1}X$ and

$$P(U,u)^{-1}X = [\nu(u,U) \cdot_U X]U + X = [P(U) + U \otimes \nu(u,U)^\flat] \llcorner X \ , \tag{4.7}$$

which gives a useful expression for the inverse projection.

This map appears in the transformation law for the electric and magnetic fields. Suppose ${}^{(4)}F^\alpha{}_\beta$ is the mixed form of the electromagnetic 2-form ${}^{(4)}F^\flat$. The electric and magnetic (vector) fields seen by u result from its measurement by u, together with the spatial duality operation in the latter case

$$E(u) = {}^{(4)}F \llcorner u \ , \quad B(u) = {}^{*(u)}P(u){}^{(4)}F^\sharp \ , \tag{4.8}$$

or in index notation

$$E(u)^\alpha = {}^{(4)}F^\alpha{}_\beta u^\beta \ , \quad B(u)^\alpha = \tfrac{1}{2}\eta(u)^{\alpha\beta\gamma}{}^{(4)}F_{\beta\gamma} \ , \tag{4.9}$$

with

$$ {}^{(4)}F^\flat = u^\flat \wedge E(u)^\flat + {}^{*(u)}B(u)^\flat \ . \tag{4.10}$$

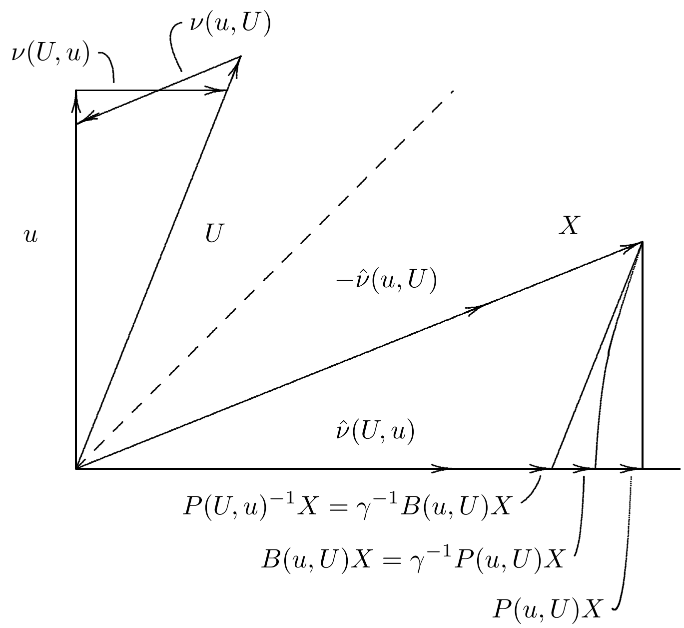

Figure 1: The relationship between the various maps on the relative observer plane of u and U. The unit vector $\hat{\nu}(U,u)$ gives the direction of the subspace belonging to the local rest space LRS_u, while $\hat{\nu}(u,U)$ does the same for LRS_U.

The transformation of the electric and magnetic fields is simple. For example, using the fact that

$$[P(U)^{(4)}F] \llcorner \nu(u,U) = \nu(u,U) \times_U B(U) \ , \tag{4.11}$$

one finds

$$\begin{aligned} P(U,u)E(u) &= \gamma P(U)\{^{(4)}F \llcorner [U + \nu(u,U)]\} \\ &= \gamma[E(U) + \nu(u,U) \times_U B(U)] \ , \end{aligned} \tag{4.12}$$

and similarly

$$P(U,u)B(u) = \gamma[B(U) - \nu(u,U) \times_U E(U)] \ . \tag{4.13}$$

Equivalently one may write

$$\begin{aligned} E(u) &= \gamma P(U,u)^{-1}[E(U) + \nu(u,U) \times_U B(U)] \ , \\ B(u) &= \gamma P(U,u)^{-1}[B(U) - \nu(u,U) \times_U E(U)] \ . \end{aligned} \tag{4.14}$$

Any map between the local rest spaces may be "measured" by one of the observers, i.e., expressed entirely in terms of quantities which are spatial with respect to that observer. For example, the mixed tensor

$$P(U,u) = P(U) \llcorner P(u) = P(U) \llcorner P(U,u) = P(U,u) \llcorner P(u) \tag{4.15}$$

(which expands to $P(u) + \gamma U \otimes \nu(U,u)$), corresponding to the linear map $P(U,u) : LRS_u \to LRS_U$, is spatial with respect to u in its covariant index and with respect to U in its contravariant index, i.e., is a "connecting tensor" in the terminology of Schouten [49]. It has associated with it two tensors

$$\begin{aligned} P(U) &= P(U,u) \llcorner P(U,u)^{-1} \ , \\ P(u) &= P(U,u)^{-1} \llcorner P(U,u) \ , \end{aligned} \tag{4.16}$$

which are spatial with respect to U and u respectively and correspond to identity transformations of each local rest space into itself. In the same way any linear map $M(U,u) : LRS_u \to LRS_U$ is represented by such a connecting tensor and has associated with it two tensors $M_U(U,u)$ and $M_u(U,u)$ which are spatial with respect to U and u respectively and act as linear transformations of the respective local rest spaces into themselves

$$\begin{aligned} M(U,u) &= M_U(U,u) \llcorner P(U,u) \\ &= P(U,u) \llcorner M_u(U,u) \ , \\ M_U(U,u) &= M(U,u) \llcorner P(U,u)^{-1} \ , \\ M_u(U,u) &= P(U,u)^{-1} \llcorner M(U,u) \ . \end{aligned} \tag{4.17}$$

These latter tensors enable one to express the map in terms of the spatial projections of just one of the observers.

The individual projections parallel and perpendicular to the direction of relative motion between the local rest spaces and within each local rest space have the representations

$$\begin{aligned} P^{(||)}(U,u) &= -\gamma \hat\nu(u,U) \otimes \hat\nu(U,u)^\flat \ , & P^{(\perp)}(U,u) &= P(U,u) - P^{(||)}(U,u) \ , \\ P_U^{(||)}(U,u) &= \hat\nu(u,U) \otimes \hat\nu(u,U)^\flat \ , & P_U^{(\perp)}(U,u) &= P(U) - P_U^{(||)}(U,u) \ , \\ P_u^{(||)}(U,u) &= \hat\nu(U,u) \otimes \hat\nu(U,u)^\flat \ , & P_u^{(\perp)}(U,u) &= P(u) - P_u^{(||)}(U,u) \ , \end{aligned} \tag{4.18}$$

where $P(U,u)\hat\nu(u,U) = -\gamma\hat\nu(U,u)$ explains the γ factor in the first relation. These in turn may be used to similarly decompose the boost $B_{(\mathrm{lrs})}(U,u)$ and the inverse projection $P(u,U)^{-1}$, for which one has the obvious relations (see Figure 1)

$$\begin{aligned} P^{(||)}(u,U)^{-1} &= \gamma^{-1} B_{(\mathrm{lrs})}^{(||)}(U,u) = \gamma^{-2} P^{(||)}(U,u) \ , \\ P^{(\perp)}(u,U)^{-1} &= B_{(\mathrm{lrs})}^{(\perp)}(U,u) = P^{(\perp)}(U,u) \end{aligned} \tag{4.19}$$

which may be used to reconstruct the spatial tensors associated with the boost and inverse projection.

For example, for the inverse boost $B_{(\mathrm{lrs})}(u,U)$ one has

$$\begin{aligned} B_{(\mathrm{lrs})\,u}(u,U) &= P(u) - \gamma(\gamma+1)^{-1} \nu(U,u) \otimes \nu(U,u)^\flat \ , \\ B_{(\mathrm{lrs})\,U}(u,U) &= P(U) - \gamma(\gamma+1)^{-1} \nu(u,U) \otimes \nu(u,U)^\flat \ , \end{aligned} \tag{4.20}$$

which follows from the expansion of

$$\begin{aligned} B_{(\mathrm{lrs})\,u}(u,U) &= B_{(\mathrm{lrs})}{}^{(\perp)}{}_u(u,U) + B^{(||)}_{(\mathrm{lrs})}(u,U)_u \\ &= P^{(\perp)}_u(u,U) + \gamma^{-1} P^{(||)}_u(u,U) \ . \end{aligned} \tag{4.21}$$

Thus if $S \in LRS_U$, then its inverse boost is

$$B_{(\mathrm{lrs})}(u,U)S = [P(u) - \gamma(\gamma+1)^{-1}\nu(U,u)\otimes\nu(U,u)^\flat] \llcorner P(u,U)S \ . \tag{4.22}$$

The map $P(u,U)P(U,u)$ is an isomorphism of LRS_u into itself which turns up in manipulations with these maps. It and its inverse have the following expressions

$$\begin{aligned} P(u,U)P(U,u) &= P^{(\perp)}_u(u) + \gamma^2 P^{(||)}_u(u,U) \\ &= P(u) + \gamma^2 \nu(U,u)\otimes\nu(U,u)^\flat \ , \\ [P(u,U)P(U,u)]^{-1} &= P(U,u)^{-1}P(u,U)^{-1} = P_u(U,u)^{-1} \\ &= P^{(\perp)}_u(U,u) + \gamma^{-2} P^{(||)}_u(U,u) \\ &= P(u) - \nu(U,u)\otimes\nu(U,u)^\flat \ , \end{aligned} \tag{4.23}$$

giving an explicit representation of the inverse projection as well.

The transformation of the electric and magnetic fields takes a more familiar form if one re-expresses it in terms of the parallel/perpendicular decomposition of the boost using Eq. (4.19)

$$\begin{aligned} E^{(||)}(u) &= B^{(||)}_{(\mathrm{lrs})}(u,U)E^{(||)}(U) \ , \\ E^{(\perp)}(u) &= \gamma B^{(\perp)}_{(\mathrm{lrs})}(u,U)[E^{(\perp)}(U) - \nu(u,U)\times_U B^{(\perp)}(U)] \ , \end{aligned} \tag{4.24}$$

with analogous expressions for the magnetic field. When expressed in a pair of orthonormal frames adapted to the two local rest spaces and related by the boost, these reduce to the familiar component expressions in a direct way.

5 Relative kinematics: derivatives

Suppose one uses the suggestive notation

$$^{(4)}D(U)/d\tau_U = {}^{(4)}\nabla_U \tag{5.1}$$

for the "total covariant derivative" along U. Its spatial projection with respect to u and rescaling corresponding to the reparametrization of Eq. (4.4) is then given by the "Fermi-Walker total spatial covariant derivative," defined by

$$\begin{aligned} D_{(\mathrm{fw})}(U,u)/d\tau_{(U,u)} &= \gamma^{-1}D_{(\mathrm{fw})}(U,u)/d\tau_U = \gamma^{-1}P(u)^{(4)}D(U)/d\tau_U \\ &= \nabla_{(\mathrm{fw})}(u) + \nabla(u)_{\nu(U,u)} \ . \end{aligned} \tag{5.2}$$

Extend this to two other similar derivative operators (the co-rotating Fermi-Walker and the Lie total spatial covariant derivatives) by

$$D_{(\text{tem})}(U,u)/d\tau_{(U,u)} = \nabla_{(\text{tem})}(u) + \nabla(u)_{\nu(U,\,u)}\ , \qquad \text{tem=fw,cfw,lie}\ , \tag{5.3}$$

which are then related to each other in the same way as the corresponding temporal derivative operators

$$\begin{aligned} D_{(\text{cfw})}(U,u)X^\alpha/d\tau_{(U,u)} &= D_{(\text{fw})}(U,u)X^\alpha/d\tau_{(U,u)} + \omega(u)^\alpha{}_\beta X^\beta \\ &= D_{(\text{lie})}(U,u)X^\alpha/d\tau_{(U,u)} + \theta(u)^\alpha{}_\beta X^\beta \end{aligned} \tag{5.4}$$

when acting on a spatial vector field X. All of these derivative operators reduce to the ordinary parameter derivative $D/d\tau_{(U,u)} \equiv d/d\tau_{(U,u)}$ when acting on a function and extend in an obvious way to all tensor fields. The co-rotating Fermi-Walker total spatial covariant derivative was introduced by Massa [8, 10].

Explicit expressions in an observer-adapted frame for these operators acting on spatial fields are easily obtained by combining Eqs. (3.8), (3.9) and (3.11). For example, if the spatial frame undergoes co-rotating Fermi-Walker transport along u, then for a spatial vector field X one finds

$$D_{(\text{cfw})}(U,u)X^a/d\tau_{(U,u)} = dX^a/d\tau_{(U,u)} + \Gamma(u)^a{}_{bc}X^c\nu(U,u)^b\ . \tag{5.5}$$

Introduce the ordinary and co-rotating Fermi-Walker and the Lie "relative accelerations" of U with respect to u by

$$a_{(\text{tem})}(U,u) = D_{(\text{tem})}(U,u)\nu(U,u)/d\tau_{(U,u)}\ , \qquad \text{tem=fw,cfw,lie}\ . \tag{5.6}$$

These are related to each other in the same way as the corresponding derivative operators in Eq. (2.18).

The total spatial covariant derivative operators restrict in a natural way to a single timelike worldline with 4-velocity U, where the $D/d\tau$ notation is most appropriate; ${}^{(4)}D(U)/d\tau_U$ is often called the absolute or intrinsic derivative along the worldline of U (associated with an induced connection along such a worldline [23]). One can then study a single worldline of a test particle with respect to the given family of test observers. One can also introduce corresponding spatial transport operations along U using these three operations by requiring that a field have zero derivative along the worldline with respect to the corresponding total spatial covariant derivative. Call these spatial ordinary and co-rotating Fermi-Walker transport and spatial Lie transport along U with respect to u, where the term "spatial" is understood to refer to u.

6 Spatial gravitational fields

The worldline of a test particle of nonzero mass m, timelike 4-velocity U and 4-acceleration $a(U)$ under the influence of a force $f(U)$ satisfies the "acceleration equals force" equation

$$a(U) = \tilde{f}(U) \equiv f(U)/m \tag{6.1}$$

which equates the 4-acceleration to the force per unit mass. The spatial projection of this equation with respect to u generalizes the more familiar spatial force equation of special relativity by the occurrence of kinematical terms which may be interpreted as "spatial gravitational forces," as well as by the presence of the spatial covariant derivative. Before projecting this equation, one must define the spatial acceleration and force analogous to their special relativistic definitions.

According to Eq. (4.6) with u and U interchanged, any vector X which is orthogonal to U has the following decomposition with respect to u

$$X = [\nu(U,u) \cdot_u P(u,U)X]u + P(u,U)X \ . \tag{6.2}$$

In particular the acceleration $a(U) = {}^{(4)}D(U)U/d\tau_U$ of the unit vector U and therefore the applied 4-force per unit mass $\tilde{f}(U)$ are orthogonal to U so

$$\begin{aligned} a(U) &= \gamma(U,u)[\nu(U,u) \cdot_u A(U,u)\, u + A(U,u)] \ , \\ \tilde{f}(U) &= \gamma(U,u)[\nu(U,u) \cdot_u \tilde{F}(U,u)\, u + \tilde{F}(U,u)] \ , \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} A(U,u) &= \gamma(U,u)^{-1} P(u,U) a(U) \ , \\ \tilde{F}(U,u) &= \gamma(U,u)^{-1} P(u,U) \tilde{f}(U) \ . \end{aligned} \tag{6.4}$$

Evaluating the rescaled spatial projection $A(U,u)$ of the acceleration by making explicit the projection of the total covariant derivative of U

$$\begin{aligned} A(U,u) &= \gamma(U,u)^{-1} P(u)^{(4)}D(U)U/d\tau_U \\ &= D_{(\mathrm{fw})}(U,u)[\gamma(U,u)u + \tilde{p}(U,u)]/d\tau_{(U,u)} \\ &= D_{(\mathrm{fw})}(U,u)\tilde{p}(U,u)/d\tau_{(U,u)} - \tilde{F}^{(\mathrm{G})}_{(\mathrm{fw})}(U,u) \ , \end{aligned} \tag{6.5}$$

where

$$\tilde{F}^{(\mathrm{G})}_{(\mathrm{fw})}(U,u) = -\gamma(U,u) D_{(\mathrm{fw})}(U,u)u/d\tau_{(U,u)} \ , \tag{6.6}$$

leads to the introduction of three analogously defined quantities

$$A(U,u) = D_{(\mathrm{tem})}(U,u)\tilde{p}(U,u)/d\tau_{(U,u)} - \tilde{F}^{(\mathrm{G})}_{(\mathrm{tem})}(U,u) \ , \quad {\scriptstyle \mathrm{tem=fw,cfw,lie}} \ , \tag{6.7}$$

which are related to each other in the same way that the corresponding total spatial covariant derivatives (and relative accelerations when $X = u$) are related to each other in Eq. (5.4). One may also express these relations in terms of the relative accelerations by substituting $\tilde{p}(U,u) = \gamma(U,u)\nu(U,u)$, leading to

$$A(U,u) = \gamma P(u,U) \llcorner P(U,u) \llcorner a_{(\mathrm{tem})}(U,u) - \tilde{F}^{(\mathrm{G})}_{(\mathrm{tem})}(U,u) \ , \quad {\scriptstyle \mathrm{tem=fw,cfw}} \ , \tag{6.8}$$

which can be inverted to yield

$$a_{(\mathrm{tem})}(U,u) = \gamma(U,u)^{-1} P_u(U,u)^{-1} \llcorner [\tilde{F}^{(\mathrm{G})}_{(\mathrm{tem})}(U,u) + A(U,u)] \ , \quad {\scriptstyle \mathrm{tem=fw,cfw}} \ , \tag{6.9}$$

where the composed projection map and the relative spatial projection tensor are given by Eq. (4.23). An additional expansion term along $\nu(U,u)$ arises in the Lie case from the derivative of $\gamma(U,u)$.

Given these definitions coming from analyzing the acceleration alone, the rescaled spatial projection $A(U,u) = \tilde{F}(U,u)$ of the force equation $a(U) = \tilde{f}(U)$ can then be expressed in the form

$$D_{(\mathrm{tem})}(U,u)\tilde{p}(U,u)/d\tau_{(U,u)} = \tilde{F}^{(\mathrm{G})}_{(\mathrm{tem})}(U,u) + \tilde{F}(U,u) \ , \quad \text{tem=fw,cfw,lie} \ , \tag{6.10}$$

leading to the identification of the terms $\tilde{F}^{(\mathrm{G})}_{(\mathrm{fw})}(U,u)$, $\tilde{F}^{(\mathrm{G})}_{(\mathrm{cfw})}(U,u)$, and $\tilde{F}^{(\mathrm{G})}_{(\mathrm{lie})}(U,u)$ respectively as the ordinary and co-rotating Fermi-Walker and the Lie spatial gravitational forces per unit mass. Since index shifting does not commute with the Lie total spatial covariant derivative, it is convenient to define also the covariant Lie spatial gravitational force $\tilde{F}^{(\mathrm{G})}_{(\mathrm{lie}\flat)}(U,u)$ per unit mass by

$$[D_{(\mathrm{lie})}(U,u)\tilde{p}(U,u)^{\flat}/d\tau_{(U,u)}]^{\sharp} = \tilde{F}^{(\mathrm{G})}_{(\mathrm{lie}\flat)}(U,u) + \tilde{F}(U,u) \ . \tag{6.11}$$

Similarly the rescaled temporal projection of the force equation yields the power equation

$$\begin{aligned} D(U,u)\tilde{E}(U,u)/d\tau_{(U,u)} &= [\tilde{F}^{(\mathrm{G})}_{(\mathrm{tem})}(U,u) + \tilde{F}(U,u)] \cdot_u \nu(U,u) \\ &\quad + \epsilon_{(\mathrm{tem})}\gamma(U,u)\theta(u)^{\flat}(\nu(U,u),\nu(U,u)) \ , \\ \epsilon_{(\mathrm{tem})} &= (0,0,1,-1) \ , \quad \text{tem=fw,cfw,lie,lie}\flat \ . \end{aligned} \tag{6.12}$$

Explicit expressions for the various spatial gravitational forces follow from their definitions (6.6), (6.7), and (6.11). The covariant Lie total spatial covariant derivative differs from the Lie total spatial covariant derivative by an expansion term arising from the commutation of the index shifting and the derivative. All of these forces have the same general form

$$\tilde{F}^{(\mathrm{G})}_{(\mathrm{tem})}(U,u) = \gamma(U,u)[\vec{g}(u) + H_{(\mathrm{tem})}(U,u) \llcorner \nu(U,u)] \ , \tag{6.13}$$

where

$$\vec{g}(u) = -a(u) \tag{6.14}$$

defines the gravitoelectric vector field $g(u)^{\alpha}$ and

$$\begin{aligned} H_{(\mathrm{fw})}(u) &= \omega(u) - \theta(u) = k(u) \ , \\ H_{(\mathrm{cfw})}(u) &= 2\omega(u) - \theta(u) \ , \\ H_{(\mathrm{lie})}(u) &= 2\omega(u) - 2\theta(u) = 2k(u) \ , \\ H_{(\mathrm{lie}\flat)}(u) &= 2\omega(u) \ , \end{aligned} \tag{6.15}$$

define the various mixed gravitomagnetic tensor fields $H_{(\mathrm{tem})}(u)^{\alpha}{}_{\beta}$ that may be introduced. If one defines a single gravitomagnetic vector field $H(u)^{\alpha}$ by

$$\vec{H}(u) = 2\vec{\omega}(u) \ , \tag{6.16}$$

then the antisymmetric part of the gravitomagnetic force contributes a term

$$\begin{aligned}\mathrm{ALT}\, H_{(\mathrm{tem})}(u) \llcorner \nu(U,u) &= \varepsilon_{(\mathrm{tem})}\, \nu(U,u) \times_u \vec{H}(u) \ , \\ \varepsilon_{(\mathrm{tem})} = (\tfrac{1}{2},1,1,1) \ , &\quad {\scriptstyle \mathrm{tem=fw,cfw,lie,lie\flat}} \ ,\end{aligned} \tag{6.17}$$

with the term in the Fermi-Walker spatial gravitational force differing by a factor of one half from those of the remaining gravitational forces.

For comparison the spatial force associated with the electromagnetic Lorentz force on a test particle of charge q and mass m due to electric and magnetic fields $E(u)$ and $B(u)$ measured by u and the corresponding spatial gravitational force are is

$$\begin{aligned} F^{(\mathrm{EM})}(u) &= q[E(u) + \nu(U,u) \times_u B(u)] \ , \\ F^{(\mathrm{G})}_{(\mathrm{tem})}(U,u) &= m\gamma(U,u)[\vec{g}(u) + \varepsilon_{(\mathrm{tem})}\, \nu(U,u) \times_u \vec{H}(u) \\ &\qquad + \mathrm{SYM}\, H_{(\mathrm{tem})}(u) \llcorner \nu(U,u)] \ , \\ &\qquad {\scriptstyle \mathrm{tem=fw,cfw,lie,lie\flat}} \ . \end{aligned} \tag{6.18}$$

Apart from the additional gamma factor in the spatial gravitational force and the symmetric tensor contribution, the close similarity of the two expressions makes the analogy with the Lorentz force and the origin of the gravitoelectromagnetic jargon clear.

The gravitomagnetic symmetric tensor field, which has no analog in electromagnetism, arises from the temporal derivative of the spatial metric, which is the new ingredient. The spatial derivatives of the spatial metric enter the total spatial covariant derivative of the spatial momentum as a "space curvature" force term

$$\begin{aligned} D_{(\mathrm{tem})}(U,u)\tilde{p}(U,u)^a/d\tau_{(U,u)} &= d\tilde{p}(U,u)^a/d\tau_{(U,u)} + C_{(\mathrm{tem})}(u)^a{}_b \tilde{p}(U,u)^b \\ &\quad + \Gamma(u)^a{}_{bc}\tilde{p}(U,u)^c \nu(U,u)^b \end{aligned} \tag{6.19}$$

which is quadratic in the spatial velocity. One can thus think of the spatial metric as a potential for these two different spatial forces. The matrix $C_{(\mathrm{tem})}(u)^a{}_b$ given by Eq. (3.11) depends on how the spatial frame is transported along the congruence and may be conveniently be chosen to vanish for one of the three temporal derivatives.

To handle the case of a lightlike test particle with zero rest mass, one must work with the null 4-momentum P instead of a 4-velocity. The substitutions

$$(U, \gamma(U,u), \tilde{p}(U,u), \nu(U,u)) \to (P, E(P,u), p(P,u), \nu(P,u)) \tag{6.20}$$

lead to analogous spatial force and power equations for this case.

7 Maxwell-like equations

The analogy between the gravitoelectromagnetic vector fields and the electromagnetic ones shows that the exterior derivative of the observer velocity 1-form

corresponds to the electromagnetic 2-form, which is itself locally the exterior derivative of a 4-potential 1-form

$$\begin{aligned} d\,{}^{(4)}A &= u^\flat \wedge E(u)^\flat + {}^{*(u)}B(u)^\flat = {}^{(4)}F^\flat \ , \\ du^\flat &= u^\flat \wedge \vec{g}(u)^\flat + {}^{*(u)}\vec{H}(u)^\flat \ . \end{aligned} \tag{7.1}$$

The observer 4-velocity thus acts as the 4-potential for the gravitoelectromagnetic vector fields.

The splitting of the identity $d^2\,{}^{(4)}A^\flat = 0$ leads to half of Maxwell's equations

$$\begin{aligned} &\mathrm{div}_u\, B(u) + \vec{H}(u) \cdot_u E(u) = 0 \ , \\ &\mathrm{curl}_u\, E(u) - \vec{g}(u) \times_u E(u) + [\pounds(u)_u + \Theta(u)]B(u) = 0 \ . \end{aligned} \tag{7.2}$$

Replacing ${}^{(4)}A$ by ${}^{(4)}u$ reduces these to the corresponding gravitoelectromagnetic equations

$$\begin{aligned} &[\mathrm{div}_u + \vec{g}(u) \cdot_u]\vec{H}(u) = 0 \ , \\ &\mathrm{curl}_u\, \vec{g}(u) + [\pounds(u)_u + \Theta(u)]\vec{H}(u) = 0 \ . \end{aligned} \tag{7.3}$$

Splitting the remaining half of Maxwell's equations

$$^*d^*\,{}^{(4)}F = 4\pi\,{}^{(4)}J \tag{7.4}$$

leads to

$$\begin{aligned} &\mathrm{div}_u\, E(u) - \vec{H}(u) \cdot_u B(u) = 4\pi\rho(u) \ , \\ &\mathrm{curl}_u\, B(u) - \vec{g}(u) \times_u B(u) - [\pounds(u)_u + \Theta(u)]E(u) = 4\pi J(u) \ , \end{aligned} \tag{7.5}$$

where ${}^{(4)}J = \rho(u)u + J(u)$ is the splitting of the 4-current.

The remaining Maxwell-like equations for the gravitoelectromagnetic vector fields arise from the Einstein equations. In order to state them one must first introduce appropriate spatial curvature tensors associated with the spatial part of the spatial connection of the observer congruence u. There are in fact four different spatial curvature tensors one may introduce [44]. Three of them have the invariant definition

$$\begin{aligned} &\big\{[\nabla(u)_X, \nabla(u)_Y] - \nabla(u)_{[X,Y]}\big\}Z \\ &\quad = R_{(\mathrm{tem})}(u)(X,Y)Z + 2\omega(u)^\flat(X,Y)\nabla_{(\mathrm{tem})}(u)Z \ , \\ &\qquad \mathrm{tem} = \mathrm{fw,cfw,lie} \ , \end{aligned} \tag{7.6}$$

where X, Y, and Z are spatial vector fields. These three tensors, the Fermi-Walker spatial curvature tensor [8], the co-rotating Fermi-Walker spatial curvature tensor [44] and the Lie spatial curvature tensor [4, 5] differ by the same kinematical terms as the temporal derivative operators themselves but reversed in sign and in a tensor product with twice the rotation tensor. In an observer-adapted frame their components are

$$\begin{aligned} R_{(\mathrm{tem})}(u)^a{}_{bcd} = {}& 2\partial_{[c}\Gamma(u)^a{}_{d]b} - C^e{}_{cd}\Gamma(u)^a{}_{eb} \\ & + 2\Gamma(u)^a{}_{[c|e|}\Gamma(u)^e{}_{d]b} - 2C_{(\mathrm{tem})}(u)^a{}_b\omega(u)_{cd} \ . \end{aligned} \tag{7.7}$$

The fourth "symmetry-obeying" spatial curvature tensor [40, 44] $R_{(\mathrm{sym})}(u)$ is related to them by the following component formulas in an observer-adapted frame

$$\begin{aligned} R_{(\mathrm{sym})}(u)^{ab}{}_{cd} &= R_{(\mathrm{lie})}(u)^{ab}{}_{cd} - 2\theta(u)^{ab}\omega(u)_{cd} - 4\theta(u)^{[a}{}_{[c}\omega(u)^{b]}{}_{d]} \\ &= R_{(\mathrm{lie})}(u)^{[ab]}{}_{cd} - 4\theta(u)^{[a}{}_{[c}\omega(u)^{b]}{}_{d]} \\ &= R_{(\mathrm{cfw})}(u)^{[ab]}{}_{cd} - 4\theta(u)^{[a}{}_{[c}\omega(u)^{b]}{}_{d]} \\ &= R_{(\mathrm{fw})}(u)^{[ab]}{}_{cd} - 2\omega(u)^{ab}\omega(u)_{cd} - 4\theta(u)^{[a}{}_{[c}\omega(u)^{b]}{}_{d]} \end{aligned} \tag{7.8}$$

and obeys all the usual symmetry identities of a 3-metric curvature tensor, as discussed by Ferrarese [40], so one may define its symmetric Ricci tensor $R_{(\mathrm{sym})}(u)^a{}_b$ and symmetric Einstein tensor $G_{(\mathrm{sym})}(u)^a{}_b$ by the usual formulas. For a hypersurface splitting in which u has vanishing vorticity, all of these spatial curvature tensors coincide with the curvature tensor of the induced Riemannian metric on the spacelike hypersurfaces orthogonal to u.

The spacetime Einstein tensor, the scalar curvature and the spacetime Ricci tensor have the following components in an observer-adapted frame in each point of view

$$\begin{aligned} 2\,{}^{(4)}G^\top{}_\top &= \mathrm{Tr}\,\theta(u)^2 - \Theta(u)^2 - \tfrac{3}{2}H(u)^cH(u)_c - R_{(\mathrm{sym})}(u)^c{}_c \ , \\ 2\,{}^{(4)}G^\top{}_a &= 2\,{}^{(4)}R^\top{}_a = -2\nabla(u)_b[\theta(u)^b{}_a - \delta^b{}_a\Theta(u)] - \{[\vec{\nabla}(u) - 2\vec{g}(u)] \times_u \vec{H}(u)\}_a \ , \\ {}^{(4)}G^a{}_b &= \{\pounds(u)_u + \Theta(u)\}[\theta(u)^a{}_b - \delta^a{}_b\Theta(u)] + \tfrac{1}{2}\delta^a{}_b[\mathrm{Tr}\,\theta(u)^2 - \Theta(u)^2] \\ &\quad + [\nabla(u)_{(b} - g(u)_{(b}]g(u)^{a)} - \delta^a{}_b[\nabla(u)_c - g(u)_c]g(u)^c \\ &\quad - \tfrac{1}{2}H(u)^aH(u)_b + \tfrac{1}{4}\delta^a{}_bH(u)^cH(u)_c + G_{(\mathrm{sym})}(u)^a{}_b \ , \\ {}^{(4)}R &= 2\{\pounds(u)_u + \Theta(u)\}\Theta(u) + \mathrm{Tr}\,\theta(u)^2 - \Theta(u)^2 + 2[\nabla(u)_a - g(u)_a]g(u)^a \\ &\quad + \tfrac{1}{2}H(u)^cH(u)_c + R_{(\mathrm{sym})}(u)^c{}_c \ , \\ {}^{(4)}R^\top{}_\top &= [\pounds(u)_u + \Theta(u)]\Theta(u) + \mathrm{Tr}\,\theta(u)^2 - \Theta(u)^2 \\ &\quad + [\nabla(u)_c - g(u)_c]g(u)^c - \tfrac{1}{2}H(u)^cH(u)_c \ , \\ {}^{(4)}R^a{}_b &= -\{\pounds(u)_u + \Theta(u)\}\theta(u)^a{}_b + [\nabla(u)_{(b} - g(u)_{(b}]g(u)^{a)} \\ &\quad - \tfrac{1}{2}[H(u)^aH(u)_b - \delta^a{}_bH(u)^cH(u)_c] + R_{(\mathrm{sym})}(u)^a{}_b \ . \end{aligned} \tag{7.9}$$

The spatial scalar and spatial vector equations which result from the measurement of the Ricci form of the Einstein equations provide the much more complicated analogs of the source driven pair of Maxwell equations

$$\begin{aligned} {}^{(4)}R^\top{}_\top &= 8\pi[{}^{(4)}T^\top{}_\top - \tfrac{1}{2}{}^{(4)}T^\alpha{}_\alpha] \ , \\ 2\,{}^{(4)}R^\top{}_a &= 16\pi\,{}^{(4)}T^\top{}_a \ , \end{aligned} \tag{7.10}$$

The complication arises from the new ingredient described by the spatial metric which has no analog in linear electrodynamics. This effect ("space curvature") also appears in the acceleration equals force equation in the total spatial covariant derivative operator.

8 Transformation of spatial gravitational fields

The spatial gravitational force fields are simply related to the kinematical quantities associated with the observer congruence. If one has two distinct observer congruences with unit tangents u and U, one can describe the transformation law between the spatial gravitational fields observed by each. One need only express the quantities and operators in the expression for the spatial gravitational fields of one in terms of those of the other to obtain such laws, as in the above derivation of the transformation law for the electric and magnetic fields.

Abbreviating $\gamma(U,u)$ to γ, the acceleration and kinematical field transform as follows

$$\begin{aligned} a(U) &= \gamma^2 P(u,U)^{-1}[a(u) - k(u) \llcorner \nu(U,u)] \\ &\quad + \gamma^2 P(U,u) a_{(\mathrm{fw})}(U,u) \ , \\ k(U) &= \gamma^2 P(U,u)[k(u) - a(u) \otimes \nu(U,u)^\flat] \\ &\quad - \gamma \nabla(U) \nu(U,u) \ . \end{aligned} \tag{8.1}$$

The rotation tensor and vector then transform as

$$\begin{aligned} \omega(U)^\flat &= \gamma^2 P(U,u)[\omega(u)^\flat - \tfrac{1}{2} a(u) \wedge \nu(U,u)] \\ &\quad + \tfrac{1}{2}\gamma d(U) \nu(U,u)^\flat \ , \\ \vec\omega(U) &= \gamma^2 P(u,U)^{-1}[\vec\omega(u) + \tfrac{1}{2}\nu(U,u) \times_u a(u)] \\ &\quad + \tfrac{1}{2}\gamma \, \mathrm{curl}_U \, \nu(U,u) \ , \end{aligned} \tag{8.2}$$

Converting to the gravitoelectromagnetic symbols leads to

$$\begin{aligned} \vec H(U) &= \gamma^2 P(u,U)^{-1}[\vec H(u) - \nu(U,u) \times_u \vec g(u)] \\ &\quad + \gamma \, \mathrm{curl}_U \, \nu(U,u) \ , \\ \vec g(U) &= \gamma^2 P(u,U)^{-1}[\vec g(u) + \tfrac{1}{2}\nu(U,u) \times_u \vec H(u) \\ &\quad - \theta(u) \llcorner \nu(U,u)] - \gamma^2 P(U,u) a_{(\mathrm{fw})}(U,u) \\ &= \gamma^2 P(u,U)^{-1}[\vec g(u) + \nu(U,u) \times_u \vec H(u) \\ &\quad - \theta(u) \llcorner \nu(U,u)] - \gamma^2 P(U,u) a_{(\mathrm{cfw})}(U,u) \ , \end{aligned} \tag{8.3}$$

where the expressions in square brackets in the gravitoelectric field transformation laws are just $\gamma^{-1}\tilde F^{(\mathrm{G})}_{(\mathrm{fw})}(U,u)$ and $\gamma^{-1}\tilde F^{(\mathrm{G})}_{(\mathrm{cfw})}(U,u)$ respectively, analogous to the Lorentz force and its magnetic analog which appear in the transformation law for the electric and magnetic fields. The terms explicitly involving the gravitoelectromagnetic vector fields in the transformation law for the gravitomagnetic vector field and in the second form of the one for the gravitoelectric vector field are exactly analogous to the corresponding transformation laws for the magnetic and electric fields, apart from the extra gamma factor also present in the force law itself.

Apart from the expansion term, the remaining part of the transformation law which breaks this correspondence, namely the relative acceleration and the relative velocity curl, can be further expanded. The relative acceleration, for

example, can be re-expressed using Eq. (6.9) with $A(U,u)$ replaced by $\tilde{F}(U,u)$. For the relative curl, one can apply the following useful formula

$$\mathrm{curl}_U\, X = \gamma P(u,U)^{-1}\{\mathrm{curl}_u\, X + \nu(U,u) \times_u [\pounds(u)_u X^\flat]^\sharp\} \ , \tag{8.4}$$

valid when X is spatial with respect to u.

For an observer-adapted co-rotating Fermi-Walker orthonormal frame, the gravitoelectric and gravitomagnetic fields are related to the 2-form which results from evaluating the tensor-valued connection 1-form on u in a way similar to the way the electric and magnetic fields are related to the electromagnetic 2-form [44]. The homogeneous part of the transformation law for the connection then leads to the terms in the transformation law for the gravitoelectric and gravitomagnetic vector fields which are analogous to those for the electric and magnetic fields.

9 Gyroscope precession

The spin vector of a test gyro carried by an observer of the observer congruence undergoes Fermi-Walker transport along u and therefore rotates relative to a co-rotating Fermi-Walker transported spatial frame with an angular velocity of "gravitomagnetic precession" given by

$$D_{(\mathrm{cfw})}(u,u)S/d\tau_u = \zeta_{(\mathrm{gm})}(u) \times_u S \ , \qquad \zeta_{(\mathrm{gm})}(u) = -\vec{\omega}(u) = -\tfrac{1}{2}\vec{H}(u) \ . \tag{9.1}$$

Along an arbitrary timelike worldline with 4-velocity U, the spin vector has the decomposition

$$S = [\nu(U,u) \cdot_u \vec{S}]u + \vec{S} \ , \quad \vec{S} \equiv P(u,U)S \tag{9.2}$$

and its length $||S|| = [S_\alpha S^\alpha]^{1/2}$ remains constant under Fermi-Walker transport. The spin vector $\vec{S}$ observed by u both rotates and changes in magnitude.

By straightforward projection of the Fermi-Walker transport equation, one finds for the observed spin vector

$$\begin{aligned} D_{(\mathrm{cfw})}(U,u)\vec{S}/d\tau_{(U,u)} \quad &= \quad -\vec{\omega}(u) \times_u \vec{S} + \gamma^{-1}[\nu(U,u) \cdot_u \vec{S}]\tilde{F}^{(\mathrm{G})}_{(\mathrm{fw})}(U,u) \\ &\qquad -\gamma[A(U,u)^\flat \llcorner P_u(U,u)^{-1} \llcorner \vec{S}]\, \nu(U,u) \ , \end{aligned} \tag{9.3}$$

where the relative spatial projection tensor is given by Eq. (4.23).

Introducing its length $||\vec{S}|| = [\vec{S} \cdot_u \vec{S}]^{1/2}$ and its direction $\hat{S} = ||\vec{S}||^{-1}\vec{S}$, one finds

$$\begin{aligned} D_{(\mathrm{cfw})}(U,u) \ln ||\vec{S}||/d\tau_{(U,u)} &= \gamma^{-1}[\nu(U,u) \cdot_u \hat{S}]\, [\tilde{F}^{(\mathrm{G})}_{(\mathrm{fw})}(U,u) \cdot_u \hat{S}] \\ &\qquad - \gamma[A(U,u)^\flat \llcorner P_u(U,u)^{-1} \llcorner \hat{S}]\, [\nu(U,u) \cdot_u \hat{S}] \ , \end{aligned} \tag{9.4}$$

and

$$D_{(\mathrm{cfw})}(U,u)\hat{S}/d\tau_{(U,u)} = \Omega_{(\mathrm{cfw})}(\hat{S},U,u) \times_u \hat{S} \ , \tag{9.5}$$

where

$$\Omega_{(\mathrm{cfw})}(\hat{S},U,u) = -\vec{\omega}(u) - \gamma^{-1}[\nu(U,u)\cdot_u \hat{S}]\,\tilde{F}^{(\mathrm{G})}_{(\mathrm{fw})}(U,u)\times_u \hat{S} \\ + \gamma[A(U,u)^\flat \llcorner P_u(U,u)^{-1} \llcorner \hat{S}]\,\nu(U,u)\times_u \hat{S}\ . \tag{9.6}$$

These formulas describe the precession of the spin vector as seen by the family of different observers of the observer congruence along the gyro's worldline.

To describe the precession as seen by the observer carrying the gyro, one must first decide with respect to what the precession will be measured. Suppose $\{e_a\}$ is an orthonormal spatial frame which is tied to the congruence, i.e., undergoes co-rotating Fermi-Walker transport along u. The observer carrying the gyro will see these axes at each event along his worldline to be in relative motion. The orientation of these moving axes with respect to the local rest space of this observer can only be defined to be the orientation of the axes which are aligned with the moving axes but momentarily at rest. (In fact, the projections $P(U,u)e_a$ into LRS_U, namely the moving frame vectors as seen by U, will no longer be orthonormal.) Thus the orientation of the spin vector S with respect to the axes $B_{(\mathrm{lrs})}(U,u)e_a$ "momentarily at rest" is well-defined and represents the orientation of S with respect to the moving axes e_a.

However, the orientation of S with respect to $B_{(\mathrm{lrs})}(U,u)e_a$ is the same as the orientation with respect to e_a of the spin vector $\mathcal{S} \equiv B_{(\mathrm{lrs})}(u,U)S$ momentarily at rest with respect to the congruence observer, since the boost is an isometry. Thus the angular velocity of the spin vector with respect to the sequence of congruence spatial frames as observed in its own local rest space equals the angular velocity of the boosted spin vector relative to the sequence of congruence frames as observed by the sequence of congruence observers, apart from a proper time renormalization.

The boosted spin vector $\mathcal{S}$ is given by Eq. (4.22) rewritten in terms of the momentum per unit mass

$$\mathcal{S} = \vec{S} - \gamma^{-1}(\gamma+1)^{-1}[\tilde{p}(U,u)\cdot_u \vec{S}]\tilde{p}(U,u) \tag{9.7}$$

and its evolution along the worldline is a simple consequence of the above result for $\vec{S}$ coupled with the spatial force equation for $\tilde{p}(U,u)$ and the power equation for $\tilde{E}(U,u)=\gamma$. The result

$$D_{(\mathrm{tem})}(U,u)\mathcal{S}/d\tau_{(U,u)} = \zeta_{(\mathrm{tem})}(U,u)\times_u \mathcal{S}\ , \qquad \mathrm{tem}=\mathrm{fw,cfw} \tag{9.8}$$

defines respectively the Fermi-Walker and co-rotating Fermi-Walker "relative angular velocities" $\zeta_{(\mathrm{fw})}(U,u)$ and

$$\zeta_{(\mathrm{cfw})}(U,u) = -\vec{\omega}(u) + \zeta_{(\mathrm{fw})}(U,u) \tag{9.9}$$

of U with respect to u. The latter one may be expressed in the form

$$\zeta_{(\mathrm{cfw})}(U,u) = -\tfrac{1}{2}\vec{H}(u) - \gamma(\gamma+1)^{-1}\nu(U,u)\times_u \tilde{F}(U,u) \\ + (\gamma+1)^{-1}\,\nu(U,u)\times_u \tilde{F}^{(\mathrm{G})}_{(\mathrm{fw})}(U,u) \\ = \zeta_{(\mathrm{gm})}(u) + \zeta_{(\mathrm{thom})}(U,u) + \zeta_{(\mathrm{geo})}(U,u) \tag{9.10}$$

in terms of the spatial projection $\tilde{F}(U,u)$ of the applied force or equivalently

$$\begin{aligned}\zeta_{(\mathrm{cfw})}(U,u) \quad = \quad & -\tfrac{1}{2}\vec{H}(u) - \gamma^2(\gamma+1)^{-1}\nu(U,u)\times_u a_{(\mathrm{cfw})}(U,u) \\ & +\nu(U,u)\times_u [\tilde{F}^{(\mathrm{G})}_{(\mathrm{cfw})}(U,u) \\ & \qquad -\gamma(\gamma+1)^{-1}\nu(U,u)\times_u \vec{\omega}(u)]\end{aligned} \tag{9.11}$$

in terms of the relative acceleration, the latter formula due to Massa and Zordan [9]. This angular velocity, in contrast with the result for $\Omega(\hat{S},U,u)$, depends only on the relative boost between the local rest spaces. One may show that in terms of the corresponding connecting tensor field, the Fermi-Walker relative angular velocity is given by the following Lie algebra type derivative expression

$$\{P(u)[{}^{(4)}D(U)\,B_{(\mathrm{lrs})}(u,U)/d\tau_{(U,u)} \llcorner B_{(\mathrm{lrs})}(U,u)]\}^{\sharp} = -{}^{*(u)}\zeta_{(\mathrm{fw})}(U,u)\ . \tag{9.12}$$

The co-rotating Fermi-Walker relative angular velocity describes how the boosted spin vector rotates with respect to an orthonormal spatial frame defined along the gyro worldline by spatial co-rotating Fermi-Walker transport (spatial in each case with respect to the observer congruence). This transport transports an orthonormal spatial frame parallel to itself except for the additional boost which keeps it spatial and the minimal rotation needed to keep it from rotating with respect to the observer congruence. However, if the worldline returns to a given observer of the observer congruence, the spatial curvature of the spatial part of the spatial connection $\nabla(u)$ will result in a net rotation compared to that observer relative to his neighbors.

If one is really interested in measuring the rotation relative to a preferred orthonormal spatial frame tied to the congruence by spatial co-rotating Fermi-Walker transport, then one must eliminate the additional rotation due to the spatial curvature. For such an orthonormal spatial frame, the expression (5.5) for the co-rotating Fermi-Walker total spatial covariant derivative along U decomposes into the ordinary derivative minus a term due to the relative rotation of a moving spatial co-rotating Fermi-Walker transported frame and the one at rest in the congruence

$$\begin{aligned}D_{(\mathrm{cfw})}(U,u)\mathcal{S}^a/d\tau_{(U,u)} &= d\mathcal{S}^a/d\tau_{(U,u)} + \Gamma(u)^a{}_{bc}\nu(U,u)^b\mathcal{S}^c \\ &= d\mathcal{S}^a/d\tau_{(U,u)} - \eta(u)^a{}_{bc}\zeta_{(\mathrm{sc})}(U,u)^b\mathcal{S}^c\ ,\end{aligned} \tag{9.13}$$

The "space curvature" angular velocity $\zeta_{(\mathrm{sc})}(U,u)$ is just the spatial dual of the value of the tensor-valued antisymmetric spatial connection 1-form evaluated along the relative velocity.

One then has the angular velocity of the boosted spin vector relative to the preferred frame as the sum of the space curvature term associated with the preferred frame and the relative angular velocity of the spin vector with respect to the observer congruence. In terms of components in such a frame one has

$$d\mathcal{S}^a/d\tau_{(U,u)} = \epsilon_{abc}[\zeta_{(\mathrm{cfw})}(U,u) + \zeta_{(\mathrm{sc})}(U,u)]^b\mathcal{S}^c\ . \tag{9.14}$$

The original angular velocity of the spin vector relative to the preferred frame as seen by the observer carrying the gyro then has the expression

$$\zeta(U,u,e) = \gamma[\zeta_{(\mathrm{cfw})}(U,u) + \zeta_{(\mathrm{sc})}(U,u)] \ , \tag{9.15}$$

taking into account the relative proper time factor.

The co-rotating Fermi-Walker relative angular velocity consists of three terms. The first term, the gravitomagnetic precession, is also referred to as the frame-dragging or Lense-Thirring [51] precession, and is independent of the relative velocity of the gyro. The last two terms together, containing explicit factors of the relative velocity, define the Fermi-Walker relative angular velocity $\zeta_{(\mathrm{fw})}(U,u)$. The first of these two terms, due to nongravitational forces (or possible Riemann tensor forces), is the Thomas precession [50], while the remaining term is called the geodetic or de Sitter or Fokker precession [52, 53, 54].

In flat spacetime with u a unit timelike Killing vector field, corresponding to time translations, the spatial gravitational force is zero and all three relative accelerations coincide with the "usual" 3-acceleration of special relativity. The second term in either formula for the co-rotating Fermi-Walker angular velocity

$$\begin{aligned}\zeta_{(\mathrm{thom})}(U,u) &= -\gamma^2(\gamma+1)^{-1}\nu(U,u)\times_u a_{(\mathrm{cfw})}(U,u)\\ &= -||\nu(U,u)||^{-2}(\gamma-1)\nu(U,u)\times_u a_{(\mathrm{cfw})}(U,u)\end{aligned} \tag{9.16}$$

is the Thomas precession due to the acceleration of the gyro. For circular motion with angular velocity $\vec{\Omega}$, the precession angular velocity is $[\gamma-1]\vec{\Omega}$, as described in exercise 6.9 of Misner, Thorne and Wheeler [30]. The first term in their equation 6.28 is exactly the boosted spin vector $\mathcal{S}$. In the limit $||\nu(U,u)|| \ll 1$ and $\gamma \to 1$ of nonrelativistic motion, the Thomas precession reduces to

$$\begin{aligned}\zeta_{(\mathrm{thom})}(U,u) &\to -\tfrac{1}{2}\nu(U,u)\times_u a_{(\mathrm{cfw})}(U,u)\\ &\to -\tfrac{1}{2}\nu(U,u)\times_u \tilde{F}(U,u) \ .\end{aligned} \tag{9.17}$$

For a geodesic in an arbitrary spacetime, the Thomas precession vanishes leaving the last term which describes the geodetic precession. In the limit of nonrelativistic motion, it reduces to

$$\zeta_{(\mathrm{so})}(U,u) = \tfrac{1}{2}\nu(U,u)\times_u \vec{g}(u) \ . \tag{9.18}$$

Thorne [14] describes this nonrelativistic term as an "induced gravitomagnetic precession" or "spin-orbit" precession since it corresponds to the gravitomagnetic precession due to an additional "induced" gravitomagnetic field $\vec{H}(u)_{(\mathrm{ind})} = -\nu(U,u)\times_u \vec{g}(u)$ induced by the motion of the gyro in the gravitoelectric field in analogy with the induced magnetic field due to motion in an electric field.

10 Spatial gravitational potentials

The action of the gravitational field on test bodies is described in the context of a partial splitting of spacetime by the spatial gravitational force fields and the

spatial part of the spatial connection, all of which together represent the spacetime connection. In a certain sense the spatial metric $P(u)^\flat$ is a potential for the spatial part of the spatial connection $\nabla(u) \circ P(u)$ and for the expansion tensor, while $u^\flat$ itself acts as a potential for the vector spatial gravitational force fields through Eq. (7.1). However, the latter relationship does not involve the spatial or temporal derivatives of spatial quantities, like the scalar and vector potentials that result from the splitting of the electromagnetic 4-potential. One needs a full splitting in order to introduce a 4-potential for the gravitoelectromagnetic vector fields in a way analogous to the electromagnetic case.

Suppose one has a parametrized nonlinear reference frame as described in the introduction. This may be specified locally by a pair (e_0, ω^0) consisting of the differential $\omega^0 = dt$ of some time function for the slicing and a vector field e_0 tangent to the threading with $\omega^0(e_0) = 1$, so that in a comoving coordinates with respect to e_0 (i.e., local coordinates $\{x^\alpha\} = \{t, x^a\}$ "adapted" to the parametrized nonlinear reference frame), e_0 has the representation $\partial/\partial t$. In the slicing and threading points of view, ω^0 and e_0 respectively are timelike, determining the temporal features of the nonlinear reference frame through the specification of the observer congruence, while the remaining element of the pair determines the choice of spatial gauge (the threading in the slicing point of view and the slicing in the threading point of view).

In the slicing point of view, the slicing 1-form is timelike and can be normalized, while in the threading point of view the threading vector field is timelike and can be normalized

$$\begin{aligned} \omega^\perp &= N\omega^0 \ , & N^{-2} &= -{}^{(4)}\mathrm{g}^{-1}(\omega^0, \omega^0) \ , \\ e_\top &= M^{-1} e_0 \equiv m \ , & M^2 &= -{}^{(4)}\mathrm{g}(e_0, e_0) \ , \end{aligned} \tag{10.1}$$

and then index-shifted and sign-reversed to define the dual object

$$e_\perp = -\omega^{\top\,\sharp} \equiv n \ , \quad \omega^\top = -e_\top{}^\flat \ . \tag{10.2}$$

The future-pointing unit normal $n \equiv e_\perp$ to the slicing and the unit tangent vector field $m \equiv e_\top$ to the threading respectively serve as the 4-velocity of the corresponding family of test observers for the two points of view, which will be referred to commonly as o. The normalization factors $L(n) = N$ and $L(m) = M$, the lapse functions in each point of view, relate the observer proper time along the observer worldlines to the parametrization associated with the parametrized nonlinear reference frame

$$d\tau_n/dt = N \ , \quad d\tau_m/dt = M \ . \tag{10.3}$$

Splitting the spatial gauge field then yields the shift vector field $\vec{N}$ and 1-form $\bar{\bar{M}}$ respectively as its spatial projection

$$\begin{aligned} e_0 &= T(n)e_0 + P(n)e_0 &&= Ne_\perp + \vec{N} \ , \\ \omega^0 &= T(m)\omega^0 + P(m)\omega^0 &&= M^{-1}\omega^\top + \bar{\bar{M}} \ . \end{aligned} \tag{10.4}$$

In slicing point of view the shift is most naturally considered as a vector field, and it determines the tilting of the threading curves away from the normal direction n. In the threading point of view the shift is most naturally considered as a 1-form, and it determines the tilting of the threading local rest spaces LRS_m away from the directions tangent to the slicing. Let $\bar{\bar{N}} = \vec{N}^\flat$ and $\vec{M} = \bar{\bar{M}}^\sharp$ denote the slicing shift 1-form and the threading shift vector field respectively. The lapse and shift terminology in the slicing point of view is due to Wheeler [55].

In the slicing point of view, $N^{-1}\vec{N}$ is the relative velocity of the threading with respect to the observer congruence, while in the threading point of view $M\vec{M}$ is the negative of the relative velocity of the direction normal to the slicing with respect to the observer congruence. When the slicing is spacelike and the threading timelike, both points of view hold and these equal the quantities $\nu(m,n)$ and $-\nu(n,m)$ respectively associated with the relative observer boost $B(m,n)$, which has gamma factor $\gamma(m,n) = N/M = d\tau_{(n,m)}/d\tau_m$. Both relative velocities (and shifts) vanish in the case of an orthogonal nonlinear reference frame where the slicing and threading points of view coincide. One can also introduce the terminology "quasi-orthogonal" for a nonlinear reference frame for which both points of view and the condition $||\nu(m,n)|| \ll 1$ are valid.

The nonlinear reference frame determines a "reference decomposition" of each tangent space into the threading subspace along e_0 (projection $\mathcal{T}$) and into the slicing subspace (projection $\mathcal{P}$) which is the kernel of ω^0. The representation of the orthogonal observer decomposition of the tangent space in terms of this (in general) nonorthogonal decomposition provides potentials for the spatial gravitational force fields. In the case in which both the slicing and threading points of view hold, the restriction of the reference spatial projection $\mathcal{P}$ to LRS_m gives the inverse map $P(m,n)^{-1}$ associated with the relative observer boost, while its restriction to the dual LRS_n^* of the slicing local rest space gives the inverse map $P(n,m)^{-1}$.

Complete the pair (e_0, ω^0) to an adapted frame $\{e_\alpha\}$ with dual frame $\{\omega^\alpha\}$, where the spatial frame $\{e_a\}$ spans the slicing subspace of the tangent space at each point. Choose the spatial frame to be comoving, i.e., Lie dragged along e_0. Then it is a computational frame as introduced by York [35], characterized by only having its spatial structure functions nonzero

$$C^\alpha{}_{\beta\gamma} = \omega^\alpha([e_\beta, e_\gamma]) = \delta^\alpha{}_a \delta^b{}_\beta \delta^c{}_\gamma C^a{}_{bc} \ . \tag{10.5}$$

Finally let $g_{ab} = P(n)_{ab}$ and $\gamma_{ab} = P(m)_{ab}$ respectively denote the spatial metric components in this frame, following the conventions of Misner, Thorne and Wheeler [30] and Landau and Lifshitz [2] respectively.

Given these definitions, the spacetime metric ${}^{(4)}\mathrm{g} = {}^{(4)}g_{\alpha\beta}\omega^\alpha \otimes \omega^\beta$ and inverse metric ${}^{(4)}\mathrm{g}^{-1} = {}^{(4)}g^{\alpha\beta} e_\alpha \otimes e_\beta$ in the computational frame in the slicing point of view are

$$\begin{aligned} {}^{(4)}\mathrm{g} &= -N^2 \omega^0 \otimes \omega^0 + g_{ab}(\omega^a + N^a \omega^0) \otimes (\omega^b + N^b \omega^0) \\ &\equiv -N^2 \omega^0 \otimes \omega^0 + g_{ab} \theta^a \otimes \theta^b \ , \end{aligned}$$

$$
\begin{aligned}
{}^{(4)}\mathrm{g}^{-1} &= -N^{-2}(e_0 - N^a e_a) \otimes (e_0 - N^b e_b) + g^{ab} e_a \otimes e_b \\
&\equiv -N^{-2} \epsilon_0 \otimes \epsilon_0 + g^{ab} e_a \otimes e_b \ ,
\end{aligned} \tag{10.6}
$$

i.e., in components

$$
\begin{aligned}
&{}^{(4)}g_{00} = -(N^2 - N_c N^c) \ , && {}^{(4)}g^{00} = -N^{-2} \ , \\
&{}^{(4)}g_{0a} = N_a \ , && {}^{(4)}g^{0a} = N^{-2} N^a \ , \\
&{}^{(4)}g_{ab} = g_{ab} \ , && {}^{(4)}g^{ab} = g^{ab} - N^{-2} N^a N^b \ ,
\end{aligned} \tag{10.7}
$$

where (g^{ab}) is the matrix inverse of the positive-definite matrix (g_{ab}). The single independent component of the volume 4-form, i.e., the (absolute value of the) square root of the metric determinant, is ${}^{(4)}g^{1/2} = N g^{1/2}$.

In the threading point of view they are instead given by

$$
\begin{aligned}
{}^{(4)}\mathrm{g} &= -M^2(\omega^0 - M_a \omega^a) \otimes (\omega^0 - M_b \omega^b) + \gamma_{ab} \omega^a \otimes \omega^b \\
&\equiv -M^2 \theta^0 \otimes \theta^0 + \gamma_{ab} \omega^a \otimes \omega^b \ , \\
{}^{(4)}\mathrm{g}^{-1} &= -M^{-2} e_0 \otimes e_0 + \gamma^{ab}(e_a + M_a e_0) \otimes (e_b + M_b e_0) \\
&\equiv -M^{-2} e_0 \otimes e_0 + \gamma^{ab} \epsilon_a \otimes \epsilon_b \ ,
\end{aligned} \tag{10.8}
$$

i.e., in components

$$
\begin{aligned}
&{}^{(4)}g_{00} = -M^2 \ , && {}^{(4)}g^{00} = -(M^{-2} - M_c M^c) \ , \\
&{}^{(4)}g_{0a} = M^2 M_a \ , && {}^{(4)}g^{0a} = M^a \ , \\
&{}^{(4)}g_{ab} = \gamma_{ab} - M^2 M_a M_b \ , && {}^{(4)}g^{ab} = \gamma^{ab} \ .
\end{aligned} \tag{10.9}
$$

Here the spatial metric matrix (γ_{ab}) is positive-definite, with inverse (γ^{ab}). Letting $\gamma = \det(\gamma_{ab}) > 0$, one has ${}^{(4)}g^{1/2} = M\gamma^{1/2}$.

In the threading point of view the projected computational frame $\{e_0, \epsilon_a\}$ with dual frame $\{\theta^0, \omega^a\}$, where $\theta^0 = T(m)\omega^0$ and $\epsilon_a = P(m)e_a$, is an observer-adapted frame which is also "spatially-comoving," namely it undergoes spatial Lie transport along the observer congruence which coincides with the threading. In the slicing point of view the projected computational frame $\{\epsilon_0, e_a\}$ with dual frame $\{\omega^0, \theta^a\}$, where $\epsilon_0 = T(n)e_0$ and $\theta^a = P(n)\omega^a$, is an observer-adapted frame, but it is not spatially-comoving along the observer congruence. Instead it undergoes spatial (with respect to n) Lie transport along the threading.

In both points of view, spatial fields have only spatially-indexed computational frame components nonzero and their indices may be shifted using the spatial metric component matrices. The reference decomposition of a tensor field corresponds to the partition of computational components according the reference "temporal" index 0 and the reference "spatial " indices $1, 2, 3$. Covariant (contravariant) spatial indices correspond to the slicing (threading) spatial projection, while contravariant (covariant) temporal indices correspond to the slicing (threading) temporal projection.

For the slicing and threading parametrizations of the spacetime metric, it is precisely the two explicit terms in the final representation of ${}^{(4)}\mathrm{g}$ and ${}^{(4)}\mathrm{g}^{-1}$

above which correspond to the covariant and contravariant form of the orthogonal projections along the local time and space directions. The spatial metric in each case is just the covariant form of the projection. In the slicing point of view, its restriction to a slice yields the induced metric on the slice submanifold making it into a Riemannian manifold, while in the threading point of view it yields the projected metric on the slice submanifold, making the slice into a different Riemannian manifold representing the projected geometry rather than the induced geometry, the latter of which is not necessarily Riemannian in the threading point of view without an additional causality assumption. In each point of view, however, this spatial metric describes the relative distances of the worldlines of nearby observers at a given coordinate time t.

The parametrized nonlinear reference frame enables one to represent the spatial tensor algebra over spacetime in a natural way in terms of the tensor algebra of time-dependent tensor fields over the "computational 3-space," namely the quotient space of the spacetime by the threading congruence. On this 3-manifold, the time-dependent spatial metric is a Riemannian metric and its connection may be related to the spatial part of the spatial connection by a difference tensor. In the slicing point of view this difference tensor is zero since the spatial metric and the spatial connection corresponds to the induced metric on each slice with its own associated connection. In the threading point of view it involves the expansion tensor of the observer congruence.

Let ${}^{(4)}\Gamma^\alpha{}_{\beta\gamma} = \omega^\alpha({}^{(4)}\nabla_{e_\beta} e_\gamma)$ be the computational components of the spacetime connection, and let $\Gamma(o)^a{}_{bc}$ be the components of the spatial part of the spatial connection in the spatial projected computational frame, as defined by Eq. (3.7). Then the components of the spatial part of the spatial connection are

$$
\begin{aligned}
\Gamma(n)_{abc} &= {}^{(4)}\Gamma_{abc} = \tfrac{1}{2}[g_{\{ab,c\}_-} + C(n)_{\{abc\}_-}] \ ,\\
\Gamma(m)_{abc} &= \gamma_{ad}\,\gamma_{be}\gamma_{cf}\,{}^{(4)}\Gamma^{def}\\
&= \tfrac{1}{2}[\gamma_{\{ab,c\}_-} + \partial_0\gamma_{\{ab}M_{c\}_-} + C(m)_{\{abc\}_-}]\\
&= \Gamma(\gamma)_{abc} + M\theta(m)_{\{ab}M_{c\}_-} \ ,
\end{aligned}
\tag{10.10}
$$

where $\Gamma(\gamma)^a{}_{bc}$ is the connection of the projected metric on the slice, corresponding to the connection of the time-dependent metric on the computational 3-space. The expansion tensor terms arise in the threading point of view since the observer-adapted spatial frame derivatives $\epsilon_a f = \partial_a f + M_a\partial_0 f$ which occur in expressing spatial components of $d(m)f$ also involve the reference time derivative.

One may introduce all of the spatial gravitational force fields associated with the observer congruence. The threading point of view is just a representation of the congruence point of view associated with m by expressing it in terms of the nonlinear reference frame, so it provides potentials for those fields. However, the slicing point of view is distinct from the hypersurface point of view associated with n since it describes evolution in terms of the threading rather than the normal congruence and so employs a temporal derivative along e_0 rather than along the observer congruence. One must introduce a corresponding Lie total spatial covariant derivative and redefine the spatial gravitational force by the

difference term. This reintroduces a gravitomagnetic force in the slicing point of view which is zero in the hypersurface point of view due to the vanishing vorticity of the normal congruence.

The Lie temporal derivatives associated with the observer congruence are

$$\begin{aligned} \nabla_{(\mathrm{lie})}(m) &= M^{-1}\pounds(m)_{e_0} \ , \\ \nabla_{(\mathrm{lie})}(n) &= \nabla_{(\mathrm{lie})}(n,e_0) - N^{-1}\pounds(n)_{\vec{N}} \ , \end{aligned} \tag{10.11}$$

where $\nabla_{(\mathrm{lie})}(n,e_0) \equiv N^{-1}\pounds(n)_{e_0}$ defines the slicing point of view Lie temporal derivative, which is used in that point of view in order to measure evolution along the threading congruence. The temporal derivatives $M\nabla_{(\mathrm{lie})}(m)$ and $N\nabla_{(\mathrm{lie})}(n,e_0)$ of a spatial tensor field on spacetime just reduce to the ordinary time derivative of the corresponding time-dependent tensor field on the computational 3-space. The kinematical tensor $k(n)$ is just the extrinsic curvature in the slicing point of view, which has the familiar form

$$k(n)_{ab} = -\theta(n)_{ab} = -\tfrac{1}{2}N^{-1}[\pounds(n)_{e_0}g_{ab} - 2\nabla(n)_{(a}N_{b)}] \tag{10.12}$$

when written in terms of the slicing Lie temporal derivative.

In the slicing point of view the various total spatial covariant derivatives all correspond to derivatives along the vector field

$$\begin{aligned} \gamma(U,n)^{-1}U &= n + \nu(U,n) \\ &= N^{-1}e_0 + [\nu(U,n) - N^{-1}\vec{N}] \ , \end{aligned} \tag{10.13}$$

the latter form of which is its reference decomposition. The slicing point of view, measuring evolution with respect to the nonlinear reference frame, implements this reference decomposition with the new temporal derivative along the temporal component

$$\begin{aligned} D_{(\mathrm{lie})}(U,n,e_0)X/d\tau_{(U,n)} &= \nabla_{(\mathrm{lie})}(n,e_0)X + \nabla(n)_{[\nu(U,n) - N^{-1}\vec{N}]}X \\ &= D_{(\mathrm{lie})}(U,n)X/d\tau_{(n,U)} - \Delta H_{(\mathrm{lie})}(n,e_0) \llcorner X \ , \end{aligned} \tag{10.14}$$

leading to the difference term

$$\begin{aligned} \Delta H_{(\mathrm{lie})}(n,e_0) \llcorner X &= N^{-1}[\nabla(n)_{\vec{N}} - \pounds(n)_{\vec{N}}]X \ , \\ [\Delta H_{(\mathrm{lie})}(n,e_0)]^\alpha{}_\beta &= N^{-1}[\nabla(n)\vec{N}]^\alpha{}_\beta = N^{-1}\nabla(n)_\beta N^\alpha \ , \end{aligned} \tag{10.15}$$

which must be added to the hypersurface Lie spatial gravitational force to obtain the slicing version.

The various Lie gravitomagnetic tensors are

$$\begin{aligned} H_{(\mathrm{lie}\flat)}(m)_{\alpha\beta} &= 2M\nabla(m)_{[\alpha}M_{\beta]} \ , & H_{(\mathrm{lie})}(m)_{\alpha\beta} &= H_{(\mathrm{lie}\flat)}(m)_{\alpha\beta} - M^{-1}\pounds(m)_{e_0}\gamma_{\alpha\beta} \ , \\ H_{(\mathrm{lie}\flat)}(n)_{\alpha\beta} &= 0 \ , & H_{(\mathrm{lie})}(n)_{\alpha\beta} &= -N^{-1}\pounds(n)_{e_0 - \vec{N}}g_{\alpha\beta} = -2\theta(n)_{\alpha\beta} \ , \\ H_{(\mathrm{lie}\flat)}(n,e_0)_{\alpha\beta} &= N^{-1}\nabla(n)_\alpha N_\beta \ , & H_{(\mathrm{lie})}(n,e_0)_{\alpha\beta} &= H_{(\mathrm{lie}\flat)}(n,e_0)_{\alpha\beta} - N^{-1}\pounds(n)_{e_0}g_{\alpha\beta} \ , \end{aligned} \tag{10.16}$$

where $g_{\alpha\beta} = P(n)_{\alpha\beta}$ and $\gamma_{\alpha\beta} = P(m)_{\alpha\beta}$ and only Latin indices are necessary when expressed in the projected computational frame. The gravitomagnetic vectors are

$$\begin{aligned} \vec{H}_{(\text{lie})}(m) &= M \operatorname{curl}_m \vec{M} \ , \\ \vec{H}_{(\text{lie})}(n) &= 0 \ , \\ \vec{H}_{(\text{lie})}(n, e_0) &= N^{-1} \operatorname{curl}_n \vec{N} \ , \end{aligned} \tag{10.17}$$

and the gravitoelectric vectors are

$$\begin{aligned} \vec{g}(m) &= -\operatorname{grad}_m(\ln M) - [\pounds(m)_{e_0} \bar{\bar{M}}]^\sharp \ , \\ \vec{g}(n) &= -\operatorname{grad}_n(\ln N) \ . \end{aligned} \tag{10.18}$$

This leads to the interpretation of

$$\Phi(o) = \ln L(o) \tag{10.19}$$

as the scalar gravitational potential and the shift as the vector gravitational potential in each point of view, together determining the vector gravitational force fields. With these definitions the slicing Lie total spatial gravitational forces are then

$$\begin{aligned} \tilde{F}^{(\text{G})}_{(\text{tem})}(U, n, e_0) = \gamma(U, n)[\vec{g}(n) + \tfrac{1}{2}\nu(U, n) \times_n \vec{H}_{(\text{lie})}(n, e_0) \\ + \operatorname{SYM} H_{(\text{tem})}(n, e_0) \llcorner \nu(U, n)] \ . \\ \text{tem=lie,lie}\flat \ , \end{aligned} \tag{10.20}$$

The threading point of view is merely a representation of the congruence point of view associated with m, so

$$m^\flat = -\omega^\top = -M[\omega^0 - \bar{\bar{M}}] \tag{10.21}$$

acts as the 4-potential of the gravitoelectromagnetic vector fields as in Eq. (7.1). A similar statement describes the hypersurface point of view, but the slicing point of view does not admit a 4-potential in this sense.

The gravitoelectric and gravitomagnetic force fields have been discussed in the black hole case in the slicing point of view using the slicing total spatial covariant derivative operator by Thorne et al [13]. Both Zel'manov [4] and Cattaneo [5]–[7] have discussed them from the threading point of view, while Landau and Lifshitz [1, 2] discuss only the stationary case in the threading point of view. Møller [3] discusses them both from his parametrization-dependent description of the threading point of view as well as for the threading point of view. All of these threading point of view discussions introduce the covariant Lie spatial gravitational forces. Massa [8] has re-expressed the Cattaneo approach in a somewhat more modern framework, using the co-rotating Fermi-Walker total spatial covariant derivative.

The gravitoelectromagnetic terminology is apparently due to Thorne and seems to have evolved from Forward's linearized discussion [15] of Møller's

threading point of view work (Forward uses the term protational for gravitomagnetic) which Forward used to draw an analogy between the electromagnetic field and the linearized gravitational field in general relativity. This generalized to the parametrized-post-Newtonian (PPN) discussion of Braginsky, Caves and Thorne [11] where the terminology "electric-type" and "magnetic-type" gravitational fields appeared, the latter of which became the "gravitomagnetic field" in a post-Newtonian general relativistic discussion of Braginsky, Polnarev and Thorne [12]. The "gravitoelectric field" and the gravitomagnetic tensor force finally appeared in the context of black holes in the slicing point of view in the book by Thorne et al [13].

Having introduced potentials for the gravitoelectric and gravitomagnetic vector fields by a choice of parametrized nonlinear reference frame, one can discuss the effect of spatial gauge transformations. In the threading point of view one can change the slicing, keeping the threading fixed, which will obviously not effect any quantitites defined only in terms of the threading decomposition. In particular the gravitoelectric and gravitomagnetic fields will be invariant, leading to a gauge freedom analogous to that of the scalar and vector potentials for the electric and magnetic fields.

The slicing point of view is a hybrid which depends both on the slicing and the threading, so changing the threading will leave invariant only those fields associated with the corresponding hypersurface point of view. The gravitomagnetic vector field will change, although the gravitoelectric field is trivially invariant since the lapse function is invariant.

Perhaps the easiest way to discuss these transformations is via an adapted coordinate system $\{t, x^a\}$. If one changes the slicing by the following change of adapted coordinates

$$t = t(t', x') \ , \quad x^a = x'^a \ , \tag{10.22}$$

then one easily finds from the new form of the metric that

$$\begin{aligned} M' &= [\partial'_0 t] M \ , \\ M'_a &= [\partial'_0 t]^{-1} [M_a - \partial'_a t]. \end{aligned} \tag{10.23}$$

The change of threading for fixed slicing is less interesting. The lapse is invariant and the shift is augmented by an additional vector field which is the difference between the old and new shift vector fields, apart from the change of spatial coordinates which is induced.

One can also consider the temporal gauge freedom associated with changing the observer congruence itself in each point of view. In the slicing (threading) point of view, this corresponds to a change of slicing (threading). The change of slicing leads to an similar tranformation of the lapse and shift as well as the spatial metric in the slicing point of view.

11 Second-order acceleration equation

In the context of a nonlinear reference frame which enables one to represent the spacetime geometry in terms of time-dependent fields on a computational

3-space, one can re-express the first order spatial acceleration equation for the spatial momentum as a second-order equation describing the evolution of the spatial coordinates along the worldline under consideration. It is exactly this equation that describes the Coriolis and centrifugal forces in rotating coordinates in flat spacetime and which is necessary to interpret the spatial force equation in the context of an adapted coordinate system in actual problems of interest.

Suppose $\{x^\alpha\} = \{t, x^a\}$ are local coordinates adapted to the parametrized nonlinear reference frame, i.e., comoving with respect to e_0, so that the coordinate frame is a computational frame, with $C^a{}_{bc} = 0$. Let $\mathcal{U}^\alpha = dx^\alpha/dt \equiv \dot{x}^\alpha$ be the coordinate components of the coordinate velocity of a worldline with 4-velocity U

$$U^\alpha = dx^\alpha/d\tau_U = \gamma(U,o)dx^\alpha/d\tau_{(o,U)} = \Gamma(U,o)\dot{x}^\alpha \ , \tag{11.1}$$

where

$$\Gamma(U,o) \equiv |\mathcal{U}_\alpha \mathcal{U}^\alpha|^{-1/2} = dt/d\tau_U \ , \tag{11.2}$$

which has the respective values

$$\begin{aligned} \Gamma(U,m) &= M^{-1}[(1 - M_a\dot{x}^a)^2 - M^{-2}\gamma_{ab}\dot{x}^a\dot{x}^b]^{-1/2} \\ &= M^{-1}\gamma(U,m)(1 - M_a\dot{x}^a)^{-1} \ , \\ \Gamma(U,n) &= N^{-1}[1 - N^{-2}g_{ab}(\dot{x}^a + N^a)(\dot{x}^b + N^b)]^{-1/2} \\ &= N^{-1}\gamma(U,n) \ , \end{aligned} \tag{11.3}$$

is Møller's coordinate gamma factor expressed in the two points of view. Its sign-reversed reciprocal is the coordinate time Lagrangian for the timelike geodesics

$$I = -\textstyle\int d\tau_U = -\int \Gamma(U,o)^{-1}\, dt \ . \tag{11.4}$$

The momenta canonically conjugate to x^a are

$$\pi_a = \begin{cases} \Gamma(U,n)g_{ab}(\dot{x}^b + N^b) \\ \Gamma(U,m)\gamma_{ab}\dot{x}^b + \tilde{\mathcal{E}}(U,m)M_a \end{cases} \tag{11.5}$$

leading to the Hamiltonian H which equals the "coordinate energy" (per unit mass) $\tilde{\mathcal{E}}(U,o) = -U_0$ which has the respective expressions

$$\begin{aligned} \tilde{\mathcal{E}}(U,m) &= \gamma(U,m)M = M^2\Gamma(U,m)(1 - M_a\dot{x}^a) \ , \\ \tilde{\mathcal{E}}(U,n) &= \gamma(U,n)N(1 - \nu(U,n) \cdot_n N^{-1}\vec{N}) \\ &= N^2\Gamma(U,n)(1 - N^{-2}N_a[\dot{x}^a + N^a]) \ . \end{aligned} \tag{11.6}$$

The rate of change of the coordinate time with respect to the observer proper time parametrization of the worldline with 4-velocity U is

$$\begin{aligned} dt/d\tau_{(o,U)} &= \frac{dt/d\tau_U}{d\tau_{(o,U)}/d\tau_U} \\ &= \Gamma(U,o)/\gamma(U,o) = \begin{cases} M^{-1}(1 - M_a\dot{x}^a)^{-1} \\ N^{-1} \end{cases} \ . \end{aligned} \tag{11.7}$$

Note that in the threading point of view, the rates of change of coordinate time with respect to the observer proper time differ on the observer worldlines and the general worldline, while they agree in the slicing point of view.

The rates of change $\dot{x}^a$ of the spatial coordinates with respect to the coordinate time t define the reference spatial components of the coordinate spatial velocity

$$\mathcal{U}(U,o)^a = \begin{cases} \Gamma(U,m)^{-1}\tilde{p}(U,m)^a \\ \Gamma(U,n)^{-1}\tilde{p}(U,n)^a - N^a \end{cases} \tag{11.8}$$

or in terms of the velocity

$$\mathcal{U}(U,o)^a = \begin{cases} M(1-M_b\dot{x}^b)\nu(U,m)^a \\ N\nu(U,n)^a - N^a \ , \end{cases} \tag{11.9}$$

which in turn define a spatial vector in each point of view

$$\mathcal{U}(U,n) = \mathcal{U}(U,n)^a e_a \ , \quad \mathcal{U}(U,m) = \mathcal{U}(U,m)^a \epsilon_a \ . \tag{11.10}$$

The projected computational frame components of the appropriate Lie total spatial covariant derivative of the spatial vector $\mathcal{U}(U,o)$ yield the corresponding second derivatives of the spatial coordinates

$$\begin{aligned} \begin{pmatrix} D_{(\mathrm{lie})}(U,m)^2 x^a/dt^2 \\ D_{(\mathrm{lie})}(U,n,e_0)^2 x^a/dt^2 \end{pmatrix} &= \begin{pmatrix} D_{(\mathrm{lie})}(U,m)\mathcal{U}(U,m)^a/dt \\ D_{(\mathrm{lie})}(U,n,e_0)\mathcal{U}(U,n)^a/dt \end{pmatrix} \\ &= \frac{d^2x^a}{dt^2} + \Gamma(o)^a{}_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt} \ . \end{aligned} \tag{11.11}$$

These may be evaluated by insertion of the relation (11.8) between spatial momentum and coordinate velocity into the spatial force Eq. (6.10).

In the threading point of view one has Møller's result [3]

$$\begin{aligned} &\Gamma(U,m)^{-1}D_{(\mathrm{lie})}(U,m)[\Gamma(U,m)\dot{x}^a]/dt \\ &\quad = M^2(1-M_b\dot{x}^b)^2\gamma(U,m)^{-1}D_{(\mathrm{lie})}(U,m)\tilde{p}(U,m)^a/d\tau_{(U,m)} \\ &\quad = (1-M_b\dot{x}^b)^2[-\operatorname{grad}_m \tfrac{1}{2}M^2 - M^2\{\pounds(m)_{e_0}\bar{\bar{M}}\}^\sharp + M^2\gamma(U,m)^{-1}\tilde{F}(U,m)]^a \\ &\qquad + (1-M_b\dot{x}^b)[M^2\mathcal{U}(U,m)\times_m \operatorname{curl}_m \vec{M} + M\,\mathrm{SYM}\,H(m)\llcorner\mathcal{U}]^a \ . \end{aligned} \tag{11.12}$$

In the slicing point of view, something more interesting happens because of the additional shift term in the coordinate relative velocity

$$\begin{aligned} &\Gamma(U,n)^{-1}D_{(\mathrm{lie})}(U,n,e_0)[\Gamma(U,n)(\dot{x}^a+N^a)]/dt \\ &\qquad = N^2\gamma(U,n)^{-1}D_{(\mathrm{lie})}(U,n,e_0)\tilde{p}(U,n)^a/d\tau_{(n,U)} \end{aligned} \tag{11.13}$$

leading to

$$
\begin{aligned}
\Gamma(U,n)^{-1} D_{(\mathrm{lie})}(U,n,e_0)[\Gamma(U,n)\dot{x}^a]/dt \\
&= [-\operatorname{grad}_n \tfrac{1}{2}N^2 - \{\pounds(n)_{e_0}\bar{\bar{N}}\}^\sharp + \vec{N} \lrcorner \vec{\nabla}(n)\bar{\bar{N}} + \mathcal{U}(U,n) \times_n \operatorname{curl}_n \vec{N} \\
&\quad - (\mathcal{U} \lrcorner \{\pounds(n)_{e_0} P(n)^\flat\})^\sharp \\
&\quad + \vec{N}\, D_{(\mathrm{lie})}(U,n,e_0) \ln \Gamma(U,n)/dt + N^2 \gamma(U,n)^{-1} F(U,n)]^a \ .
\end{aligned}
\tag{11.14}
$$

The additional shift spatial covariant derivative term combines with the existing term in the spatial gravitational force to form twice its antisymmetric part, eliminating the contribution of the symmetric part of the shift tensor gravitomagnetic force term and doubling the contribution of the gravitomagnetic vector force to yield a term exactly analogous to the threading point of view expression, modulo a term quadratic in the shift vector field (which is the Coriolis term in the case of flat spacetime in rotating coordinates). A shift Lie derivative term also adds to the gravitoelectric field to form an expression analogous to the one in the threading point of view. The spatial metric Lie derivative term is analogous to the symmetric part of the threading gravitomagnetic tensor. An annoying correction factor for the coordinate time parametrization scales the right hand side in the threading point of view. Multiplying this out leads to higher order terms in the coordinate velocity appearing in the threading point of view expression.

In both cases the logarithmic gravitoelectric potential $\Phi(o) = \ln L(o)$ maps onto the potential $\frac{1}{2}L(o)^2 \sim \frac{1}{2}(L(o)^2 - 1)$ in the coordinate time representation of the second order acceleration equation, giving a linear rather than a logarithmic relationship between the square of the lapse and the potential. Both of these are the same in the Newtonian limit and agree with the more commonly used reference splitting definition of Møller [3]

$$
\Phi_{(\mathrm{ref})} = \tfrac{1}{2}(-{}^{(4)}g_{00} - 1) \tag{11.15}
$$

but differ at post-Newtonian order.

The changes to the slicing point of view acceleration equation which occur when switching to the second-order form are not surprising since its expression must result from a transformation of the threading point of view expressions, and in the quasi-orthogonal limit the lapse and shift of the two points of view coincide. Thorne et al [13] discuss these changes in the slicing point of view second-order acceleration equation for black hole spacetimes in the weak field slow motion limit. Forward [15] briefly discusses a reference decomposition of the second-order acceleration equation before linearizing to go to the same limit for an isolated body.

The transformation law relating the gravitoelectric and gravitomagnetic vector fields in the two points of view may be evaluated either directly from their definitions, re-expressing each in terms of the other point of view, or by applying the general congruence point of view transformation law for the relative observer boost between n and m, making the nonhomogeneous terms explicit.

The result is

$$
\begin{aligned}
\vec{g}(n) &= \gamma^2 P(n,m)\{\vec{g}(m) + [\pounds(m)_{e_0}\bar{\bar{\vec{M}}}]^\sharp + \tfrac{1}{2}\nu(n,m)\times_m \vec{H}(m) \\
&\quad + \tfrac{1}{2}\nu(n,m)\,\lrcorner M\,\mathrm{SYM}\,\vec{\nabla}(m)\bar{\bar{\vec{M}}} \\
&\quad - \gamma^{-2}M^{-1}\nu(n,m)\pounds(m)_{e_0}\ln(M\gamma)\}\ , \\
\tfrac{1}{2}\vec{H}(n,e_0) &= P(m,n)^{-1}\{\tfrac{1}{2}\vec{H}(m) - \nu(n,m)\times_m \Big[\vec{g}(m) + \tfrac{1}{2}[\pounds(m)_{e_0}\bar{\bar{\vec{M}}}]^\sharp\Big]\}\ ,
\end{aligned}
\tag{11.16}
$$

where $\gamma = \gamma(n,m) = N/M$, or the inverse transformation

$$
\begin{aligned}
\vec{g}(m) &= \gamma^2 P(n,m)^{-1}\{\vec{g}(n) - \gamma^2 N^{-2}[\pounds(n)_{e_0}\bar{\bar{\vec{N}}}]^\sharp + \tfrac{1}{2}\nu(m,n)\times_n \vec{H}(n) \\
&\quad + \nu(m,n)\,\lrcorner N^{-1}\,\mathrm{SYM}\,\vec{\nabla}(n)\bar{\bar{\vec{N}}} \\
&\quad - \gamma^{-2}N^{-1}\nu(m,n)\pounds(n)_{e_0}\ln(\gamma N^{-1})\}\ , \\
\tfrac{1}{2}\vec{H}(m) &= \gamma^2 P(n,m)^{-1}\{\tfrac{1}{2}\vec{H}(n,e_0) - \nu(m,n)\times_n \Big[\vec{g}(n) - \tfrac{1}{2}N^{-1}[\pounds(n)_{e_0}\,N^{-1}\bar{\bar{\vec{N}}}]^\sharp \\
&\quad + [\nu(m,n)\,\lrcorner\mathrm{SYM}\,N^{-1}\nabla(n)\bar{\bar{\vec{N}}}]^\sharp\Big]\}\ .
\end{aligned}
\tag{11.17}
$$

12 Stationary spacetimes and Fermat's principle

Stationary spacetimes admit a timelike Killing vector field on some open submanifold. Choosing the threading vector field e_0 to be such a Killing vector field leads to a "stationary" parametrized nonlinear reference frame whose parametrization is adapted to the flow of this vector field. The threading point of view is valid everywhere that it is timelike.

In the computational frame, all stationary fields will have components which are independent of t, and the spatial fields which result from the measurement of a stationary spacetime field reduce to time-independent fields on the computational 3-space. Spatial differential operators reduce to the obvious time-independent operators there as well, with $\pounds(m)_{e_0}$ reducing to the ordinary time (parameter) derivative d/dt. In other words as described by Gerosh [43], the algebra of stationary spatial fields is isomorphic to the tensor algebra on the computational 3-space with the Riemannian geometry of the time-independent projected spatial metric, expressable as $\gamma_{ab}dx^a \otimes dx^b$ in local adapted coordinates identified with their projections down to the computational 3-space. The spatial operators $\mathrm{grad}_m, \mathrm{curl}_m, \mathrm{div}_m$ of stationary spatial fields reduce to the corresponding operators defined with respect to this metric $(\mathrm{grad}(\gamma), \mathrm{curl}(\gamma), \mathrm{div}(\gamma)$ defined in terms of its connection $\nabla(\gamma))$ when projected down to the computational 3-space, as introduced by Landau and Lifshitz [1, 2].

The expansion tensor

$$\theta(m)_{\alpha\beta} = \tfrac{1}{2}M^{-1}[P(m)\pounds_{e_0}{}^{(4)}g]_{\alpha\beta} = 0 \tag{12.1}$$

vanishes, leading to the agreement of the two temporal operators $\nabla_{(\mathrm{cfw})}(m)$ and $\nabla_{(\mathrm{lie})}(m)$ when acting on spatial fields, while the acceleration admits a potential in the ordinary sense

$$a(m)^\flat = d(m) \ln M = d \ln M \ . \tag{12.2}$$

The vorticity vector field

$$\vec{\omega}(m) = \tfrac{1}{2} M \,\mathrm{curl}_m \vec{M} \tag{12.3}$$

reduces to the ordinary curl of the shift vector field when projected down to the computational 3-space, corrected by the lapse function. Thus the gravitoelectric and gravitomagnetic fields in the threading point of view admit scalar and vector potentials on the computational 3-space in the ordinary sense. In the static case where the vorticity vanishes, one may choose the slicing so that the shift is zero, leading to a static nonlinear reference frame. If the slicing is also spacelike, then the slicing point of view holds and one can repeat the discussion for the corresponding quantities with some differences.

For both points of view introduce the conformally rescaled spatial metric, or "optical metric"

$$\begin{aligned} \tilde{P}(o)^\flat &= L(o)^{-2} P(o)^\flat \ , \\ \tilde{\gamma}_{ab} &= M^{-2} \gamma_{ab} \ , \quad \tilde{g}_{ab} = N^{-2} g_{ab} \ . \end{aligned} \tag{12.4}$$

This re-definition of the spatial metric variable makes the square of the lapse function an overall conformal factor of the spacetime metric, which is relevant to the conformally invariant null geodesic problem for stationary spacetimes.

Møller [3] has shown that the coordinate light travel time may be used as a parametrization independent action integral for this problem, giving a general relativistic generalization of Fermat's principle. Since the differential of spacetime arclength vanishes along a null geodesic

$$- {}^{(4)}ds^2 = d\tau_{(o)}{}^2 - d\ell_{(o)}{}^2 = 0 \ , \tag{12.5}$$

using an obvious notation for its splitting in each point of view, one may solve this for dt (choosing the future-directed root), leading to the action integral between two fixed points of the computational 3-space. For the threading point of view one finds

$$\Delta t = \int dt = \int [M^{-1} d\ell_{(m)} + M_a dx^a] \tag{12.6}$$

or equivalently

$$\Delta t = \int n_{(\mathrm{refr})(m)} d\ell_{(m)} = \int \tilde{n}_{(\mathrm{refr})(m)} d\tilde{\ell}_{(m)} \ , \tag{12.7}$$

where

$$\begin{aligned} n_{(\mathrm{refr})(m)} &= M^{-1}(1 + M M_a dx^a / d\ell_m) \ , \\ \tilde{n}_{(\mathrm{refr})(m)} &= (1 + M_a dx^a / d\tilde{\ell}_m) \ , \end{aligned} \tag{12.8}$$

and $\ell_{(m)}$ and $\tilde{\ell}_{(m)}$ are the spatial arclength parameters with respect to the threading spatial metric and optical spatial metric respectively on the computational 3-space. Re-expressing these same quantities in terms of the slicing variables leads to more complicated expressions

$$\Delta t = \int n_{(\mathrm{refr})(n)} d\ell_{(n)} = \int \tilde{n}_{(\mathrm{refr})(n)} d\tilde{\ell}_{(n)} \ , \tag{12.9}$$

where

$$\begin{aligned} n_{(\mathrm{refr})(n)} &= N^{-1}[1 - N^{-2} N_b N^b]^{-1} \{[1 - N^{-2} N_b N^b \\ &\quad + (N^{-1} N_a dx^a / d\ell_{(n)})^2]^{1/2} + N_a dx^a / d\ell_{(n)}\} \ , \\ \tilde{n}_{(\mathrm{refr})(n)} &= [1 - \tilde{g}^{ab} N_a N_b]^{-1} \{[1 - \tilde{g}^{ab} N_a N_b + (N_a dx^a / d\tilde{\ell}_{(n)})^2]^{1/2} \\ &\quad + N_a dx^a / d\tilde{\ell}_{(n)}\} \ . \end{aligned} \tag{12.10}$$

These may be interpreted in two ways. For example, in the static case the action integral reduces to the arclength of the curve in the computational 3-space with respect to the optical metric, i.e., null geodesics project down to geodesics of the optical geometry, using the terminology recently introduced by Abramowicz et al [56]–[59]. Perlick has introduced the alternate name "Fermat metric" for the optical metric [60, 61, 62]. Alternatively one may interpret the lapse as introducing an effective index of refraction as discussed by Møller [3]. In the nonstatic case this index of refraction becomes anisotropic, deflecting the paths of light rays from the geodesics of the optical geometry.

For the static case using a static parametrized nonlinear reference frame, one can also re-express the acceleration equation in terms of the optical geometry. If one introduces the "coordinate momentum" (per unit mass) components $\tilde{\mathcal{P}}(U,m)^a = \tilde{\mathcal{E}}(U,m)\dot{x}^a$, then the canonical momenta are obtained by lowering the index with the optical metric $\pi_a = \tilde{\gamma}_{ab}\tilde{\mathcal{P}}(U,m)^b$. The coordinate energy $\tilde{\mathcal{E}}(U,m)$ is conserved in the stationary case since the Lagrangian is independent of t, and it is related to the coordinate momentum by

$$\tilde{\mathcal{E}}(U,m)^2 = M^2 \tilde{m}^2 + \tilde{\gamma}_{ab} \tilde{\mathcal{P}}(U,m)^a \tilde{\mathcal{P}}(U,m)^b \ , \tag{12.11}$$

where $\tilde{m}$ is the "mass per unit mass", i.e., 1 for a timelike curve and 0 for a null curve, for which this relation may be used to define the coordinate energy.

Introduce also the corresponding optical derivative using the optical spatial connection $\tilde{\nabla}(\gamma)$ instead of the spatial metric connection $\nabla(\gamma)$ on the computational 3-space. Then the static case second order acceleration equation takes the form

$$\begin{aligned} \tilde{D}_{(\mathrm{lie})}(U,m)^2 x^a / dt^2 &= \tilde{\mathcal{E}}(U,m)^{-1} \tilde{D}_{(\mathrm{lie})}(U,m)[\tilde{\mathcal{E}}(U,m)\dot{x}^a]/dt \\ &= -(\tilde{m}/\tilde{\mathcal{E}}(U,m))^2 \, \tilde{\nabla}(m)^a \tfrac{1}{2}[M^2 - 1] \\ &\quad + M/\mathcal{E}(U,m)\tilde{\gamma}^{ab} F(U,m)_b \ . \end{aligned} \tag{12.12}$$

For zero rest mass $\tilde{m} = 0$ and no applied force, this reduces to the geodesic equation for the optical geometry with the coordinate time as an affine parameter, describing null geodesics as discussed in exercise 40.3 of Misner, Thorne

and Wheeler [30]. The above null geodesic action integral is just the optical arclength function for this static case, first studied by Weyl [64].

In fact the general action integral (12.6) is parametrization independent. If λ denotes a parameter, and $\dot{f} = df/d\lambda$, then this action is

$$\int [d\tilde{\ell}_{(m)} + \bar{\bar{M}}] = \int [(\tilde{\gamma}_{ab}\dot{x}^a\dot{x}^b)^{1/2} + M_a\dot{x}^a]\, d\lambda \ . \tag{12.13}$$

For the optical arclength parametrization $d\tilde{\ell}_{(m)}/d\lambda = 1$ or $\tilde{\gamma}_{ab}\dot{x}^a\dot{x}^b = 1$, one can use instead the equivalent action

$$\int [\tfrac{1}{2}\tilde{\gamma}_{ab}\dot{x}^a\dot{x}^b + M_a\dot{x}^a]\, d\lambda \ . \tag{12.14}$$

Perlick [61] has noted that each of these actions continue to be valid in the nonstationary case, but then of course cannot be considered without the evolution equation for t as well. Clearly since the null geodesics are conformally invariant, it is enough to have a conformally stationary spacetime for the problem to reduce to a purely spatial one, allowing the same analysis to extend to interesting cosmological spacetimes [62]. One can repeat the discussion for the force equation for timelike test particles for the case of lightlike test particles and obtain the general second-order equation and the spatial force equation for that case and also develop the Lagrangian approach to the former case. This has not been done here for reasons of space.

Note that the shift plays a role similar to the effect of the vector potential in electromagnetism. Samuel and Iyer [65] and Perlick [62, 63] have explored the analogy between the gravitomagnetic field and the magnetic field for stationary spacetimes.

13 Post-Newtonian approximation

The post-Newtonian treatment of weak gravitational fields within general relativity and its parametrized post-Newtonian (PPN) generalization are based on a preferred class of local coordinate systems defined by certain functional conditions on the metric components. These preferred coordinates $\{x^\alpha\} = \{t, x^a\}$ naturally introduce the structure of a "post-Newtonian" nonlinear reference frame and a set of gauge transformations among different choices of such frames. Since these nonlinear reference frames are "quasi-orthogonal," both the slicing and threading points of view are valid and the slicing and threading variables are closely related.

Assuming the notation of Misner, Thorne and Wheeler [30], the usual post-Newtonian conditions on the coordinate components of the metric in coordinates adapted to a post-Newtonian nonlinear reference frame are the following, using the abbreviated notation $O(n) \equiv O(\epsilon^n)$

$$\begin{aligned} {}^{(4)}g_{00} &= -1 - 2\Phi + O(4) = -M^2 \ , \\ {}^{(4)}g_{0a} &= \Phi_a + O(5) = N_a \ , \\ {}^{(4)}g_{ab} &= \delta_{ab} + O(2) = g_{ab} \ , \end{aligned} \tag{13.1}$$

and

$$
\begin{aligned}
{}^{(4)}g^{00} &= -1 + 2\Phi + O(4) = -N^{-2} \ , \\
{}^{(4)}g^{0a} &= \delta^{ab}\Phi_b + O(5) = M^a \ , \\
{}^{(4)}g^{ab} &= \delta^{ab} + O(2) = \gamma^{ab} \ ,
\end{aligned}
\tag{13.2}
$$

where $\Phi \sim O(2)$ and $\Phi_a \sim O(3)$. The metric components $({}^{(4)}g_{00}, {}^{(4)}g_{0a}, {}^{(4)}g_{ab})$ are respectively (even,odd,even) in order and are cut off at orders $(O(4), O(5), O(2))$; the same is true respectively of the lapse, shift and spatial metric components in each point of view. This leaves only the $O(4)$ term in ${}^{(4)}g_{00}$ and the order $O(2)$ terms in ${}^{(4)}g_{ab}$ to make explicit.

Because of the relationships

$$
\begin{aligned}
N &= M[1 - M^2 M_a M_b \gamma^{ab}]^{-1/2} \ , \\
N_a &= M^2 M_a \ , \\
g_{ab} &= \gamma_{ab} + M^2 M_a M_b \ ,
\end{aligned}
\tag{13.3}
$$

the lapses and the spatial metric components (even order) agree through $O(4)$, while the shift components agree through order $O(3)$, both with each other and with the obvious corresponding variables defined using the reference decomposition of either the spacetime metric or inverse metric. In other words the slicing and threading metric variables agree up to the first post-Newtonian order and effectively reduce to the corresponding reference variables. The projected computational spatial frame vectors and 1-forms in the two points of view differ from the computational ones by

$$
\epsilon_a - e_a \sim O(3)\partial_0 \ , \qquad \theta^a - \omega^a \sim O(3)dt \tag{13.4}
$$

so the distinction between them is lost in the shift and spatial metric fields. The relative velocity satisfies $||\nu(m,n)|| \sim O(3)$, leading to the quasi-orthogonal condition on the nonlinear reference frame and the agreement of the slicing and threading metric variables.

Blanchet and Damour [70] fix the definition of the potential Φ to $O(4)$ by the condition that it coincide with the fully nonlinear gravitational potential

$$
\Phi = \ln L(o) + O(6) \ , \tag{13.5}
$$

or

$$
L(o) = e^{\Phi} + O(6) = 1 + \Phi + \tfrac{1}{2}\Phi^2 + O(6) \ , \tag{13.6}
$$

which is the same to the first post-Newtonian order in both points of view, so that

$$
-\,{}^{(4)}g_{00} = e^{2\Phi} + O(6) = 1 + 2\Phi + 2\Phi^2 + O(6) \ . \tag{13.7}
$$

The gravitoelectric and gravitomagnetic fields then have the following behavior

$$
\begin{aligned}
H(m)^a &= H^a + O(5) = H(n, e_0)^a + O(5) \ , \\
g(m)_a &= g_a + O(6) \ , \\
g(n)_a &= -\partial_a \Phi + O(6) \ ,
\end{aligned}
\tag{13.8}
$$

where the lowest order post-Newtonian threading fields are defined by

$$H^a = \epsilon^{abc}\partial_b\Phi_c \ , \quad g_a = -\partial_a\Phi - \partial_0\Phi_a \ , \tag{13.9}$$

The gravitomagnetic fields agree to first post-Newtonian order but the gravitoelectric fields differ by an $O(4)$ time derivative term. For some reason independent of which point of view people favor for the full Einstein equations, it is the post-Newtonian threading fields which are always used without comment [11, 17, 71], and which agree with Forward's reference decomposition of the gravitational variables [15].

In the post-Newtonian approximation the 4-potential for the threading gravitoelectromagnetic vector fields is just

$$\begin{aligned} m^\flat &= -M(dt - M_a dx^a) \\ &\to -dt + [(-\Phi + O(4))dt + (\Phi_a + O(5)dx^a] \ . \end{aligned} \tag{13.10}$$

The explicit terms in the square bracketed expression define the post-Newtonian 4-potential introduced by Damour et al [17].

The threading spatial gauge freedom to change the slicing reduces to the usual electromagnetic gauge transformations of the scalar and vector potentials (Φ, Φ_a)

$$\begin{aligned} t &\mapsto t + \Lambda + O(5) \ , \quad \Lambda \sim O(3) \ , \\ \Phi &\mapsto \Phi + \partial_0\Lambda + O(6) \ , \\ \Phi_a &\mapsto \Phi_a + \partial_a\Lambda + O(5) \ . \end{aligned} \tag{13.11}$$

This leaves the threading gravitoelectric and gravitomagnetic fields invariant.

Of the kinematical fields, only the expansion tensor remains to be evaluated and its form depends on the spatial metric. It is of order $O(3)$, differing in the two points of view at that order by spatial derivatives of the shift. To get a handle on it, one must examine the post-Newtonian restrictions on the spatial metric.

Introduce the "anti-optical" spatial metric in both points of view by

$$\begin{aligned} \tilde{P}(o)^\flat &= L(o)^2 P(o)^\flat \ , \\ \tilde{\gamma}_{ab} &= M^2\gamma_{ab} \ , \quad \tilde{g}_{ab} = N^2 g_{ab} \ . \end{aligned} \tag{13.12}$$

To post-Newtonian order the distinction between slicing and threading is unimportant for the spatial metric. The threading "anti-optical" spatial metric is associated with the generalized Lewis-Papapetrou form of the spacetime metric [66]–[68] advocated by Perjés [69]; it naturally arises in the analysis of the initial value problem in that point of view, which is the problem to which the Einstein equations reduce in the stationary case [68]. Blanchet and Damour [70, 71] have noticed that the threading anti-optical metric plays a privileged role in the post-Newtonian analysis. Damour et al [17, 72] explicitly define this metric by the relation

$$\tilde{\gamma}^{1/2}\tilde{\gamma}^{ab} = M\gamma^{1/2}\gamma^{ab} = {}^{(4)}g^{1/2\,(4)}g^{ab} \tag{13.13}$$

satisfied by the threading anti-optical metric, and they explain its importance using the simple expression for the Einstein tensor written in terms of ${}^{(4)}g^{1/2\,(4)}g^{\alpha\beta}$.

This may also be seen directly from the slicing/threading expression for the Einstein tensor. Since the projected computational frame is an observer-adapted frame, setting $u = o$ in Eq. (7.9) provides the relevant formulas. To post-Newtonian order, all of the spatial curvature tensors agree with the slicing one which is the usual curvature of the induced metric on the slice, since the differences are of order $O(6)$ [44]. The lowest order terms in the spatial projection of the spacetime Einstein tensor in the slicing point of view are

$$G_{(\mathrm{sym})}(n)^a{}_b - \nabla(n)_b a(n)^a + \delta^a{}_b \nabla(n)_c a(n)^c \sim O(2) \ . \tag{13.14}$$

Under the conformal transformation which defines the slicing anti-optical metric, the Einstein tensor has the following transformation law [68]

$$\tilde{G}(n)_{ab} = G(n)_{ab} - [\nabla(n)_a - a(n)_a]a(n)_b + g_{ab}\nabla(n)^c a(n)_c \ . \tag{13.15}$$

Thus to the lowest post-Newtonian order $O(2)$ (neglecting the terms quadratic in $a(n)$), the spatial projection of the spacetime Einstein tensor is just the Einstein tensor of the anti-optical metric, which must vanish to that order since the spatial projection of the energy-momentum tensor is of order $O(4)$. The anti-optical metric (slicing or threading) is therefore flat to order $O(2)$. This is the key observation of Damour et al [17], who choose the obvious gauge condition that the spatial coordinates be Cartesian with respect to the anti-optical metric to that order

$$\tilde{\gamma}_{ab} = \delta_{ab} + O(4) \tag{13.16}$$

or equivalently

$$\gamma_{ab} = M^{-2}\delta_{ab} + O(4) = [1 - 2\Phi]\delta_{ab} + O(4) \ . \tag{13.17}$$

The operations $\cdot$, $\times$ and $\vec{\nabla}$ will denote the flat space operations in these special coordinates. The post-Newtonian threading gravito-vector fields then have the definition

$$\vec{H} = \vec{\nabla} \times \vec{\Phi} \ , \quad \vec{g} = -\vec{\nabla}\Phi - \partial_0 \vec{\Phi} \ , \tag{13.18}$$

which have the consequences

$$\vec{\nabla} \cdot \vec{H} = 0 = \vec{\nabla} \times \vec{g} + \partial_0 \vec{H} \ . \tag{13.19}$$

These are just the post-Newtonian limit of Eqs. (7.3).

The expansion tensor then has the behavior

$$\begin{aligned} \theta(m)_{ab} &= -\partial_0\Phi\,\delta_{ab} + O(5) \ , \\ \Theta(m) &= -3\partial_0\Phi + O(5) \ . \\ \theta(n)_{ab} &= -\partial_0\Phi\,\delta_{ab} - \nabla_{(a}\Phi_{b)} + O(5) \ , \\ \Theta(n) &= -3\partial_0\Phi - \vec{\nabla}\cdot\vec{\Phi} + O(5) \ . \end{aligned} \tag{13.20}$$

The remaining Einstein equations under these conditions resemble the remaining half of Maxwell's equations. The Ricci form of the Einstein equations with the trace-reversal of the energy-momentum tensor on the right-hand side is more convenient to obtain these equations

$$ {}^{(4)}R^{\alpha\beta} = 8\pi[{}^{(4)}T^{(\mathrm{TR})}]^{\alpha\beta} = 8\pi[{}^{(4)}T^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta\,(4)}T^{\gamma}{}_{\gamma}] \ . \tag{13.21} $$

The remaining two linearly independent Einstein equations in the post-Newtonian approximation expressed in terms of the projected computational frame

$$ \begin{aligned} {}^{(4)}R^{\top\top} &= -\vec{\nabla}\cdot\vec{g} + 3{\partial_0}^2\Phi + O(6) = 8\pi[{}^{(4)}T^{(\mathrm{TR})}]^{\top\top} \ , \\ 2{}^{(4)}R^{\top}{}_{a} &= [-\vec{\nabla}\times\vec{H} + 4\partial_0\vec{g}]_a + O(5) = 16\pi[{}^{(4)}T^{(\mathrm{TR})}]^{\top}{}_{a} \ . \end{aligned} \tag{13.22} $$

are the second pair of Maxwell-like equations for the gravitoelectromagnetic vector fields given by Braginsky et al [11] and Damour et al [17].

The "standard post-Newtonian gauge condition" for the time coordinate t comes from identifying a time derivative in the $O(4)$ behavior of ${}^{(4)}R^{\top\top}$

$$ {}^{(4)}R^{\top\top} = \nabla^2\Phi + \partial_0[\vec{\nabla}\cdot\vec{\Phi} + 3\partial_0\Phi + O(5)] \ . \tag{13.23} $$

Setting the expression in square brackets to zero gives this condition, and its imposition leads to a simple Poisson equation for the scalar potential.

The harmonic gauge condition for t has the following representations

$$ 0 = {}^{(4)}\nabla_\alpha\nabla^\alpha t = \partial_\alpha[{}^{(4)}g^{1/2\,(4)}g^{0\alpha}] = \begin{cases} M\gamma^{1/2}[\mathrm{div}_m\,\vec{M} - \vec{g}(m)\cdot_m\vec{M} \\ \quad - M^{-1}\pounds(m)_m\ln(\gamma^{1/2}M^{-1})] \\ N^{-1}g^{1/2}[\mathrm{div}_n\,\vec{N} - \vec{g}(n)\cdot_n\vec{N} \\ \quad - N^{-2}\pounds(n)_{e_0}\ln(g^{1/2}N^{-1})] \ , \end{cases} \tag{13.24} $$

with the post-Newtonian limit

$$ \partial_0\Phi + \vec{\nabla}\cdot\vec{\Phi} + O(5) = 0 \ . \tag{13.25} $$

This differs only by a numerical factor from the standard gauge condition.

14 Schiff precession formula

The discussion of gyroscope precession presented above is valid for any spacetime and expresses the angular velocity of the relative rotation of the spin vector with respect to the observer congruence or subsequently with respect to a preferred orthonormal observer-adapted frame whose spatial part undergoes co-rotating Fermi-Walker transport along the observer congruence. The spatial distribution of the orientation of such a spatial orthonormal frame is still arbitrary. For the first result to be meaningful one must have a preferred observer congruence,

and for the second, a preferred distribution of the orientation of the spatial orthonormal frame.

Stationary spacetimes have a preferred observer congruence associated with any timelike Killing vector field, leading to a stationary observer congruence with a stationary 4-velocity u. Since spatial Lie transport along such a congruence coincides with co-rotating Fermi-Walker transport, the spatial projection of any frame which is comoving with respect to u will yield a spatial frame which is spatially comoving and which undergoes co-rotating Fermi-Walker transport along u. In particular comoving coordinates whose spatial coordinates are orthogonal yield such a frame under spatial projection which can be normalized to an observer-adapted orthonormal frame with the same properties.

For a stationary spacetime representing an isolated mass distribution which is asymptotically flat at spatial infinity one can pick out a preferred Killing vector (in the event of additional symmetry), namely the one which reduces to the unit vorticity-free timelike Killing vector of the asymptotic geometry with respect to which the isolated body is not moving. This leads to the static observer congruence. The choice of a spatial orthonormal frame is less clear. A "Cartesian-like" frame would be preferable but no canonical choice exists. In the post-Newtonian theory or its parametrized generalization, one works with a class of "Cartesian-like" coordinates involving a gauge freedom, so such a frame is available, modulo these gauge transformations. Nester has recently shown that a preferred slicing orthonormal frame exists which asymptotically approaches a given spatial Cartesian frame on an asymptotically flat slice [73, 74]. Its boost could be taken to define a preferred "Cartesian-like" threading orthonormal frame if one has a preferred slicing within the class of asymptotically flat nonlinear reference frames.

For black hole spacetimes, and indeed the wider class of stationary axially symmetric spacetimes, a preferred class of stationary orthonormal spatial frames does exist in both the slicing and threading points of view. Consider a black hole spacetime in the Boyer-Lindquist coordinate system, with its associated nonlinear reference frame. The threading point of view holds outside the ergosphere where the Killing observers follow the timelike time lines, while the slicing point of view holds outside of the event horizon where the slicing is spacelike. The stationary threading observers have the interpretation of being nonrotating with respect to the asymptotically flat region of spacetime and are called the static observers, while the nonstationary slicing observers have the interpretation of being locally nonrotating with respect to the spacetime geometry.

The Boyer-Lindquist spatial coordinates $\{r, \theta, \phi\}$ are orthogonal so both the coordinate derivatives $\{e_a\}$ and coordinate differentials $\{\omega^a\}$ are orthogonal and can be normalized and then completed uniquely to an (axially symmetric stationary) orthonormal spacetime frame or dual frame. Normalizing the spatial coordinate derivatives leads to the slicing orthonormal frame $\{n, e_{\hat{a}}\}$ with dual frame $\{\omega^\perp, \theta^{\hat{a}}\}$ while normalizing the spatial coordinate differentials leads to the threading orthonormal frame $\{m, \epsilon_{\hat{a}}\}$ with dual frame $\{\omega^\top, \omega^{\hat{a}}\}$.

One can boost each of these two orthonormal frames uniquely to align them

with the 4-velocity of an arbitrary gyro worldline

$$
\begin{aligned}
B(u,n)\{n,e_{\hat a}\} &= \{u,E_{(\mathrm{sl})a}\}\ ,\\
B(u,m)\{m,\epsilon_{\hat a}\} &= \{u,E_{(\mathrm{th})a}\} = B(u,m)B(m,n)\{n,e_{\hat a}\}\ .
\end{aligned} \tag{14.1}
$$

The two orthonormal frames so obtained are related to each other by the time-dependent Thomas rotation determined by the composition of the two boosts $B(u,m)$ and $B(m,n)$

$$
E_{(\mathrm{th})a} = B(u,m)B(m,n)B(n,u)E_{(\mathrm{sl})a} = R(u,m,n)E_{(\mathrm{sl})a}\ , \tag{14.2}
$$

which may in some sense be interpreted as the relative rotation of the spatial axes of the slicing and threading observers as determined by the gyro. The boosted frame in each point of view is the spatial frame that an observer following the worldline of the gyro would reconstruct as the frame he would see if that frame were not moving relative to him.

In the slicing point of view, the co-rotating Fermi-Walker relative angular velocity measures the precession of the spin relative to the locally nonrotating observers, while in the threading point of view, it is instead relative to the static Killing observers which in some sense reflect the properties of the nonrotating frame of the "distant stars" (whose incoming light rays have fixed direction with respect to a co-rotating Fermi-Walker transported frame along these observers' worldlines). However, the spatial frames described above are "spherical" in nature rather than Cartesian so the space curvature precession also includes the rotation of the observer frame relative to Cartesian-like frames along the gyro worldline.

All asymptotically flat axially symmetric stationary spacetimes have such a preferred stationary nonlinear reference frame whose slicing is orthogonal to the locally nonrotating observers and whose threading is along the static observers [26]. A similar situation exists in the PPN theory, where the PPN spatial coordinates are orthogonal to the lowest nontrivial order as well as Cartesian-like, so one can introduce a preferred class of slicing and threading orthonormal frames. The orthonormal threading frame is given in section 39.10 of Misner, Thorne and Wheeler [30].

The classic spin precession formula of Schiff [75] describes how the spin vector precesses relative to the "distant stars" as seen by an geodesic observer carrying the gyro in the gravitational field of an isolated body. It may be obtained by evaluating in the post-Newtonian order the formula (9.15) for the angular velocity $\zeta(U,u,e)$ relative to the threading orthonormal frame. One only need evaluate the space curvature precession term to this order.

The key difference with the limiting expression for $\zeta_{(\mathrm{cfw})}$ is the fact that in this limit within general relativity, the space curvature precession has twice the value of the spin orbit precession, leading to a total coefficient of $\frac{3}{2}$. It has been generalized to the PPN theory as discussed by Misner, Thorne and Wheeler [30] or Weinberg [76].

To post-Newtonian order, the slicing spatial orthonormal frame is $e_a = (1+\Phi)\partial/\partial x^a$, so the spatial structure functions, the spatial connection components

and the space curvature precession in that point of view are

$$\begin{aligned} C^a{}_{bc} &= 2\delta^a{}_{[b} g(n)_{c]} + O(4)\ , \\ \Gamma(n)_{abc} &= 2\delta_{b[a} g(n)_{c]} + O(4)\ , \\ \zeta_{(\mathrm{sc})}(U,n,e)^a &= \eta(n)^{abc}\nu(U,n)_b g(n)_c + O(5) \\ &= [\nu(U,n) \times_n \vec{g}(n)]^a + O(5) \\ &= [\nu(U,m) \times_m \vec{g}(m)]^a + O(5)\ . \end{aligned} \tag{14.3}$$

Thus in this limit the space curvature precession, which has a completely different origin from the spin-orbit precession, has twice the magnitude of the latter precession, leading to the total factor $\frac{3}{2}$

$$\begin{aligned} \zeta_{(\mathrm{gyro})}(U,m,e) \quad \to \quad & -\tfrac{1}{2}\vec{H}(m) - \tfrac{1}{2}\nu(U,m) \times_m a_{(\mathrm{cfw})}(U,m) \\ & +\tfrac{3}{2}\nu(U,m) \times_m \vec{g}(m)\ . \end{aligned} \tag{14.4}$$

The actual Schiff formula is obtained by substituting explicit expressions for the post-Newtonian gravitoelectromagnetic potentials. Its verification is the goal of the long awaited Stanford gyroscopic precession experiment [77] and of the proposed LAGEOS experiment [78]. These and other experiments [12, 79, 80] have provided much of the motivation for talking about "gravitomagnetism."

15 Conclusions

A single framework has been introduced which encompasses all possible approaches to space-plus-time splittings of spacetime and allows transformations between them to be considered. Precise relative observer maps and differential operators have been introduced which first determine the definition of spatial gravitational forces and then neatly characterize the gyro precession formula in terms of them. In the post-Newtonian approximation, all of these various spatial gravitational forces are closely related, but it is the threading forces which are universally used in the application of that approximation. By examining the origin of the post-Newtonian equations in the fully nonlinear context of a parametrized nonlinear reference frame, a better understanding of their structure is obtained.

This same scheme can be used in studying the Sagnac effect [81, 82] and the closely related synchronization gap [83], Maxwell's equations for the electromagnetic field [21, 32, 33, 84], and the fully nonlinear Einstein equations and their initial value problem. The initial value problem for the threading point of view is still not well understood [41, 42], although it is closely related to the exact solutions work for stationary spacetimes [68]. The perturbation problem for Friedmann-Robertson-Walker models has been considered both in the slicing [16, 85] and the more general congruence [19, 86, 87] points of view; the present formalism allows one to more easily relate the two. Similarly the idea of a Newtonian limit [88, 89] crucially relies on a family of spacetime splittings, for which the present language is rather helpful in describing.

Rotation in general relativity has intrigued people for quite some time, but some rather simple rotational aspects of familiar exact solutions have still not been clearly presented. Examination of black hole spacetimes, the Gödel spacetime [90] and Minkowski spacetime in a uniformly rotating nonlinear reference frame using the present approach leads to a more intuitive understanding of the familiar properties of these models and how they compare to each other in terms of the individual contributions of gravitoelectric, gravitomagnetic and space curvature effects in their various representations. This will be discussed in a subsequent article.

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Corrections

This reformatted version contains the following misprint corrections of the original article (to which the page numbers refer) and one reference publication update:

- p. 3, Section II, first sentence, remove: $= -{}^{(4)}\eta^{0123}$.
- p. 5, Eq. (2.10) both lines, after first equal sign:
  projections on both indices inserted.
- p. 8, end of phrase preceding Eq. (3.4), superscript before comma should be b.
- p. 16, Eq. (6.12), put tilde over E on left hand side of equation, first line.
- p. 17, Eq. (6.15), change (U, u) to (u) on all left hand sides.
- p. 17, Eq. (6.18), Line 1, change $F^{(\mathrm{EM})}(u)$ to $F^{(\mathrm{EM})}(U, u)$; line 3, change $H_{(\mathrm{tem})}(U, u)$ to $H_{(\mathrm{tem})}(u)$.
- p. 19, Eq. (7.8), fourth line added for $R_{(\mathrm{fw})}(u)^{[ab]}{}_{cd}$.
- p. 21, Eq. (8.1), change $\gamma P(U, u)$ to $\gamma^2 P(U, u)$ on right hand side of second equation.
- p. 23, Eq. (9.9), change $\vec{\omega}(U, u)$ to $\vec{\omega}(u)$.
- p. 28, Eq. (10.9), Line 1, second equation, left hand side, change ${}^{(4)}g^0$ to ${}^{(4)}g^{00}$.
- p. 30, Eq. (10.14), left hand side subscript before equal sign, change $d\tau_{(n,U)}$ to $d\tau_{(U,n)}$.
- p. 31, Eq. (10.16), Line 2, second equation, change
  $$H_{(\mathrm{lie})}(n)_{\alpha\beta} = -N^{-1}\pounds(n)_{e_0} g_{\alpha\beta}$$
  to
  $$H_{(\mathrm{lie})}(n)_{\alpha\beta} = -N^{-1}\pounds(n)_{e_0 - \vec{N}} g_{\alpha\beta} = -2\theta(n)_{\alpha\beta}$$
- p. 31, Eq. (10.20), change $F^{(\mathrm{G})}_{(\mathrm{tem})}(U, n, e_0)$ to $\tilde{F}^{(\mathrm{G})}_{(\mathrm{tem})}(U, n, e_0)$.
- p. 34, Eq. (11.8), change o to m and n respectively in right hand side cases expressions.
- p. 35, Eq. (11.12), Line 2, change $d\tau_{(m,U)}$ to $d\tau_{(U,m)}$.
- p. 36, Eq. (11.16), Line 3, change $(M\gamma_L)$ to $(M\gamma)$.
- p. 40, End of page, change $(O(4), O(3), O(2))$ to $(O(4), O(5), O(2))$.
- p. 42, Eq. (13.20), Lines 1 and 3, change ∂_{ab} to δ_{ab}.
- p. 50, Reference 73, change year from (1991) to (1989).